Math 6080. Fall 2014. Polynomials.

3. The Projective Plane.

"Boss, boss, ze plane, ze plane!"

–Tattoo, Fantasy Island

The **projective plane** completes the plane with points at infinity. Projective transformations of the projective plane contain the standard transformations of the plane (reflections, translations, rotations, etc) and include new and mysterious transformations that, for example, can change ellipses to hyperbolas or parabolas. Projective transformations also allow us to merge the study of conics with linear algebra by viewing a quadratic polynomial as a symmetric matrix.

As a warm-up, consider the projective line.

Definition. The **projective line** $\mathbb{RP}^1$ is the set of all lines through the origin in the plane $\mathbb{R}^2$.

First Remark. Every line through the origin intersects the *unit sphere* in exactly two points. The unit sphere in the plane is more commonly referred to as the *unit circle*:

$$S^1 := \{(x, y) \mid x^2 + y^2 = 1\}$$

The fact that every line meets the circle in two (antipodal) points means that there is a two-to-one surjective mapping:

$$a : S^1 \to \mathbb{RP}^1$$

The topological manifold $\mathbb{RP}^1$ is easy to describe. It is also a circle. In fact, the circle is the only connected, one-dimensional closed manifold. In this case, walking along $\mathbb{RP}^1$ is just the same as walking along S^1, except that you return to the start after π radians, rather than after 2π radians.

Second Remark: Ratios give coordinates for the elements of $\mathbb{RP}^1$:

$$(x : y) \neq (0 : 0) \text{ with } (x : y) = (\alpha x : \alpha y) \text{ for all } \alpha \neq 0$$

Ratios behave as fractions, except that the ratio $(x : 0)$ is allowed, while fractions do not allow 0 to be in the denominator.

Only the "ratio" $(0 : 0)$ is not permitted.

The Line Plus a Point at Infinity. The ratios $(x:1)$ can be thought of as a point of the line $y=1$. In fact, **all but one** of the ratios can be put in this form *uniquely* via:

$$(x:y) = \left(\frac{x}{y}:1\right)$$

which means that we may think of $\mathbb{RP}^1$ as *being* the points of the line $y=1$ with an added point at infinity, which corresponds to the line $y=0$, which, in ratio coordinates, is $(x:0)$ (the non-fraction). Notice that approaching infinity either with large numbers or with small numbers has **the same limit**. In other words, this point at infinity functions as both ∞ and $-\infty$ in the sense of calculus. Walking beyond infinity in the projective line brings you back to the other side!

Projective Transformations of $\mathbb{RP}^1$ are 2×2 invertible matrices:

$$A:\mathbb{RP}^1 \to \mathbb{RP}^1;\quad A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\cdot\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

This is well-defined because matrix multiplication takes lines through the origin to lines through the origin. It is **different** from a linear transformation of the plane because we are thinking of our vectors as being ratios, so scaling just acts as the identity:

$$\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha x \\ \alpha y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Recall that an affine linear transformation of the real line is:

$$f(x) = ax+b$$

where b is the translation, and a is the scaling factor. Consider the projective transformation of the projective line:

$$A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

For points of $\mathbb{R}$ (with coordinates $(x:1)$) this is the **same** as F:

$$A\left(\begin{bmatrix} x \\ 1 \end{bmatrix}\right) = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} ax+b \\ 1 \end{bmatrix}$$

and the additional point at infinity is fixed:

$$A\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

In other words, the affine transformations of the line are examples of projective transformations of the projective line.

For Discussion.

(a) Suppose $x_1 \in \mathbb{R}$. Show that there is a unique translation of the number-line (i.e. a self-mapping of the form $f(x) = x + b$) that takes x_1 to the origin $0 \in \mathbb{R}$. What is it?

(b) Suppose $x_1, x_2 \in \mathbb{R}$ are two distinct numbers. Show that there is a unique affine transformation (i.e. a slef-map of the form $f(x) = ax+b$) taking x_1 to 0 and x_1 to 1. What is it?

(c) Suppose $x_1, x_2, x_3 \in \mathbb{R}$ are three distinct points. Show that there is a unique *projective* transformation of $\mathbb{RP}^1$ taking $(x_1 : 1)$ to $(0 : 1)$, $(x_2 : 1)$ to $(1 : 1)$ and $(x_3 : 1)$ to the infinite point $(1 : 0)$. What is it?

Definition. The **projective plane** $\mathbb{RP}^2$ is the set of lines through the origin in $\mathbb{R}^3$.

Each such line meets the unit sphere:

$$S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$$

in two (antipodal) points, so there is a two-to-one surjective mapping:

$$a : S^2 \to \mathbb{RP}^2$$

Ratios. Elements of the projective plane are ratios of triples:

$$(x : y : z) \neq (0 : 0 : 0) \text{ with } (x : y : z) = (\alpha x : \alpha y : \alpha z) \text{ for } \alpha \neq 0$$

We can identify most of the elements of $\mathbb{RP}^2$ with points of the *plane* $z = 1$ via:

$$(x : y : z) = \left(\frac{x}{z} : \frac{y}{z} : 1\right)$$

and the rest of the elements represent lines in the plane $z = 0$, which is the **projective line** $\mathbb{RP}^1$ (with an added irrelevant 0) $(x : y : 0)$.

In other words, the projective plane:

$$\mathbb{RP}^2 = \mathbb{R}^2 \cup \mathbb{RP}^1$$

is the ordinary plane, plus a projective line of "points at infinity."

Example: (a) Each line $y = mx + b$ of points in the plane $\mathbb{R}^2$ meets the points at infinity in a **single** point:

$$(x : mx : 0) = (1 : m : 0)$$

This is because the projective coordinates of the "finite" points of the line (with the exception of the y-intercept) are:

$$(x : mx + b : 1) = (1 : m + b/x : 1/x)$$

and the limit as $x \to \pm\infty$ is the point $(1 : m : 0)$. This means that following the line in one direction to infinity will "flip" to the opposite end of the line. The closure of this line is a copy of $\mathbb{RP}^1$!

Remark. The point at infinity **only** depends upon the slope of the line. Thus any two lines with the same slope, i.e. parallel lines, will **meet** at the point $(1 : m : 0)$ at infinity. And two lines with different slopes, which meet inside the plane $\mathbb{R}^2$, will not meet at infinity. Thus any two lines in the real projective plane meet in a single point!

(b) Consider the region in the plane bounded above by:

$$\max\{y = -x, y = 1, y = x\}$$

and bounded below by

$$\min\{y = x, y = -1, y = -x\}$$

The points at infinity to add to this bow-tie-shaped region are:

$$\{(1 : m : 0) \mid -1 \leq m \leq 1\}$$

If you were to walk along the positive x-axis (growing linearly taller in proportion to your distance from the origin), you would reach infinity on the right and then *flip upside down* as you return from infinity on the left. This means that the closure of the bow-tie is a copy of the **Möbius strip**, which is a "one-sided" surface that cannot be embedded in a sphere. Moreover, the complementary region to the bow-tie closure is a topological disk, realizing the projective plane as a Möbius strip capped with a disk, which cannot be placed into $\mathbb{R}^3$. In particular, $\mathbb{RP}^2$ is **not** a sphere, but rather a one-sided pseudo-sphere.

Take-Home. Consider the vertices of a regular polygon in the plane. Between any two vertices there are **two** line segments, namely the ordinary "finite" line segment in the plane and the "infinite" line segment that joins the vertices through the point at infinity. For your consideration, try to count the simple polygons with your given vertices, if you allow yourself to use either finite or infinite line segments. In other words, count the ways of drawing a path of line segments that visits each vertex once and does not cross itself. For example:

Given the vertices of an equilateral triangle, **each** segment may be either finite or infinite, and the result will be a non-crossing path that traverses each vertex. Thus, there are $8 = 2^3$ possible such triangles.

There are 15 quadrilaterals(!) How many pentagons are there?