

Complex Algebraic Geometry: Smooth Curves

Aaron Bertram, 2010

9. Differentials. The differentials on an affine variety are elements of a module over the coordinate ring. Differentials localize, and in particular the germ of differentials at a nonsingular point $x \in X$ is a free module over \mathcal{O}_x of rank equal to the dimension of X . The vector space $\Omega[C]$ of differentials on a smooth projective curve C will be identified with $L(K_C)$, where K_C is called a “canonical” divisor (though it is only the linear series $|K_C|$ that is, in fact, canonical).

Let $X = V(\mathcal{P}) \subset \mathbb{C}^n$ be an affine variety, with

$$\mathcal{P} = \langle f_1, \dots, f_m \rangle \subset \mathbb{C}[x_1, \dots, x_n]$$

Definition. The module of differentials on X is:

$$\Omega[X] := \left(\bigoplus \mathbb{C}[X] dx_i \right) / \left\langle \sum \frac{\partial f_i}{\partial x_j} dx_j \right\rangle$$

This module satisfies a universal property:

Definition: A *derivation* $d : A \rightarrow M$, mapping a \mathbb{C} -algebra A to an A -module M , is a linear map that also satisfies

$$d(aa') = a(da') + a'(da) \text{ and } dc = 0 \text{ for all } c \in \mathbb{C}$$

Remark: If $\alpha : M \rightarrow N$ is a homomorphism of A -modules, then:

$$d \mapsto \alpha \circ d \text{ maps } M\text{-derivations to } N\text{-derivations}$$

Definition: $d : A \rightarrow \Omega_A$ is a *universal derivation* if all other derivations $\delta : A \rightarrow M$ are obtained by composing with a (unique) homomorphism:

$$\alpha_\delta : \Omega_A \rightarrow M$$

Exercise 9.1: (a) (General nonsense) Universal derivations always exist, and are determined up to (unique) isomorphism.

(b) The module $\Omega[X]$ is isomorphic to $\Omega_{\mathbb{C}[X]}$.

Definition: Suppose $A = \mathbb{C}[X]$ for an affine variety, and $S \subset A$ is a multiplicatively closed set not containing zero (e.g. the complement of a maximal ideal or the powers of an element of $f \in A$). Then the “localization” of A at S :

$$A_S := \left\{ \frac{a}{s} \mid a \in A, s \in S \right\} / \left(\frac{a}{s} \sim \frac{a'}{s'} \Leftrightarrow as' = a's \right) \subset \mathbb{C}(X)$$

has universal derivation given by:

$$\Omega_{A_S} = \left\{ \frac{m}{s} \mid m \in \Omega_A, s \in S \right\} / \left(\frac{m}{s} \sim \frac{m'}{s'} \Leftrightarrow t(ms' - m's) = 0 \text{ for some } t \in S \right)$$

with derivation determined by the “quotient rule:”

$$d_S : A_S \rightarrow \Omega_{A_S}; \quad d\left(\frac{a}{s}\right) := \frac{s(da) - a(ds)}{s^2}$$

Example: The universal derivation for the field $\mathbb{C}(X)$ is:

$$d : \mathbb{C}(X) \rightarrow \Omega(X)$$

where $\Omega(X)$ is a vector space over $\mathbb{C}(X)$ of dimension equal to the dimension of X .

Proposition 8.2: Let $d_x : \mathcal{O}_x \rightarrow \Omega_x$ be the universal derivation. Then:

$$\mathrm{Hom}_{\mathcal{O}_x}(\Omega_x, \mathcal{O}_x/m_x) \cong \mathrm{Hom}_{\mathcal{O}_x}(m_x/m_x^2, \mathcal{O}_x/m_x)$$

as modules (vector spaces) over $\mathbb{C} = \mathcal{O}_x/m_x$.

Proof: By the universal property,

$$\mathrm{Hom}_{\mathcal{O}_x}(\Omega_x, \mathcal{O}_x/m_x) = \{\text{derivations } d : \mathcal{O}_x \rightarrow \mathcal{O}_x/m_x\}$$

and such a derivation determines a map $\bar{d} : m_x \rightarrow \mathcal{O}_x/m_x$, which, by virtue of the Leibniz rule, vanishes on m_x^2 . Conversely, $\delta : m_x/m_x^2 \rightarrow \mathcal{O}_x/m_x$ determines a map $\bar{\delta} : m_x \rightarrow \mathcal{O}_x/m_x$, which in turn defines a derivation via $d(a) := \bar{\delta}(a - a(x))$, where $a(x) \in \mathbb{C} = \mathcal{O}_x/m_x$ is the “evaluation” of a at x .

Corollary 8.3: If $x \in X$ is a nonsingular point, then the \mathcal{O}_x -module Ω_x is free of rank equal to the dimension of X , and there exists an affine neighborhood $x \in U$ such that $\Omega_{\mathbb{C}[U]}$ is a free $\mathbb{C}[U]$ module.

Proof: Nakayama’s lemma.

Definition: Suppose X is a nonsingular variety.

- (a) A rational differential form on X is an element $\omega \in \Omega(X)$.
- (b) The regular differential forms on $U \subset X$ are the intersection:

$$\Omega[U] = \bigcap_{x \in U} \Omega_x \subset \mathbb{C}(X)$$

Remark: The freeness of Ω_x from Corollary 8.3 implies that the natural \mathcal{O}_x -module homomorphism $\Omega_x \rightarrow \Omega(X)$ is injective, which is what permits us to make the (simple) definition in (b) of a regular differential form. In general, i.e. at singular points of X , it is not true that Ω_x sits inside $\Omega(X)$, and we will need the full machinery of coherent sheaves to understand how a regular differential ought to be defined.

Rational differentials are easy to produce. For example, given any non-constant $\phi \in \mathbb{C}(X)$, then:

$$d\phi \in \Omega(X)$$

is a rational differential form. But it has poles, in general.

Example: Consider $x \in \mathbb{C}(\mathbb{P}^1)$, where we identify $\mathbb{C}[\mathbb{P}^1] = \mathbb{C}[x, y]$. Then $\Omega[\mathbb{C}^1] = \mathbb{C}[x]dx$, but using $y^{-1} = x$, we have:

$$dx = \frac{dy}{y^2}$$

from which it follows that dx has a pole of order 2 at ∞ . In fact, we will see that there are **no** regular differential forms on \mathbb{P}^1 .

More Interesting Example: Consider the affine elliptic curve:

$$E = V(y^2 - (x - r_1)(x - r_2)(x - r_3)) \subset \mathbb{C}^2$$

for distinct $r_1, r_2, r_3 \in \mathbb{C}$, and the differential form:

$$\omega := \frac{dx}{y}$$

We claim first of all that $\omega \in \Omega[E]$. Indeed, consider that:

$$2ydy = [(x - r_2)(x - r_3) + (x - r_1)(x - r_3) + (x - r_2)(x - r_3)] dx$$

from which it follows that either:

- (a) $y \neq 0$, so y is a unit and x is a uniformizing parameter, or else:
- (b) $y = 0$, and we use the alternate expression:

$$\frac{dx}{y} = \frac{dy}{[(x - r_2)(x - r_3) + (x - r_1)(x - r_3) + (x - r_2)(x - r_3)]}$$

which again has the form: $d\pi/u$.

Moreover: The form ω extends to a regular differential form on $\Omega[\overline{E}]$, where $\overline{E} \subset \mathbb{P}^2$ is the (smooth) closure of E in the projective plane. (Check this!) Thus, for example, E is not isomorphic to \mathbb{P}^1 .

Let C be a smooth projective curve, and $\omega \in \Omega(C)$ be a rational differential form. Then we may write:

$$\omega = \phi d\pi$$

where π is a uniformizing parameter at x , and:

$$\text{multiplicity at } x \text{ of } \omega := \text{multiplicity at } x \text{ of } \phi$$

and then:

$$\text{div}(\omega) := \sum_{x \in C} (\text{multiplicity at } x \text{ of } \omega) \cdot x$$

Thus, by definition, $\text{div}(\omega)$ is effective if and only if ω is regular.

Remark: This is a *finite* sum since it follows from Corollary 8.3 that if π is a uniformizing parameter at x , then $\pi - \pi(y)$ is a uniformizing parameter at each $y \in U$ of an open neighborhood of x , and then $d\pi = d(\pi - \pi(y))$, so that in fact the multiplicities of ω and of ϕ above are the same for all points of U (but they will be different, of course, for the finite set of points in the complement of U).