Complex Algebraic Geometry: Smooth Curves

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8. Curve Basics. The ring $\mathcal{O}_x$ of germs of regular functions a point at $x \in C$ of a smooth curve is a *discrete valuation ring*. This means that the maximal ideal $m_x \subset \mathcal{O}_x$ is principal, generated by a "uniformizing parameter" π_x, which then has the property that **every**

$$0 \neq \phi \in \mathbb{C}(X) \text{ can be written uniquely in the form } \phi = u_x \pi_x^{d_x}$$

where $u_x \in \mathcal{O}_x$ is a unit and $d \in \mathbb{Z}$. The integer d is the *multiplicity* of ϕ at x, and we will explore the consequences of this structure.

A rational function $\phi \in \mathbb{C}(X)$ determines a regular map:

$$\phi : U \to \mathbb{C}^1$$

on its domain of definition. We may also think of this as a map:

$$\Phi : U \to \mathbb{P}^1;\ \Phi(x) = (\phi(x) : 1)$$

to the projective line, in which case:

Proposition 8.1. Φ extends uniquely to a regular map $\Phi : C \to \mathbb{P}^1$.

Proof: The complement $C - U$ consists of the points of C where the multiplicity of ϕ is negative. Let $x \in C$ be one such point, and write $\phi = u_x \pi_x^{-d}$. Then wherever π_x and ϕ are both defined, we have:

$$(\phi : 1) = (\phi \cdot \pi_x^d : \pi_x^d)$$

as maps to $\mathbb{P}^1$. But the second expression is defined at $x \in C$, because both coordinates have non-negative multiplicity, and the first coordinate function is **not zero** at x. This extends the map across $x \in C$, and, similarly, the map extends across all the rest of the points in the complement of U.

Exercise 8.1: Extend Proposition 8.1 to prove that every regular map from an open subset $U \subset C$ to $\mathbb{P}^n$ determined by a set of rational functions $\phi_1, \dots, \phi_n \in \mathcal{O}_C(U)$:

$$\Phi = (\phi_1 : \cdots : \phi_n : 1) : U \to \mathbb{P}^n$$

extends uniquely to a regular map defined on C, and conclude that the same is true when $\mathbb{P}^n$ is replaced by any projective variety.

Definition: A *divisor* D on a smooth curve C is a finite linear combination (with integer coefficients):

$$D = \sum_{i=1}^{n} d_i x_i$$

The *degree* of D is $\deg(D) := \sum d_i$, and D is *effective* if each $d_i \geq 0$.

Definition: The divisor of zeroes of $0 \neq \phi \in \mathbb{C}(C)$ is:

$$\operatorname{div}(\phi) := \sum (\textit{muliplicity of } \phi \textit{ at } x_i) \cdot x_i$$

Proposition 8.2: If C is a smooth, *projective* curve, then:

$$\deg(\operatorname{div}(\phi)) = 0$$

for all (nonzero) rational functions $\phi \in \mathbb{C}(C)$.

Proof: Each $\phi \in \mathbb{C}(C)$ determines a regular map:

$$\Phi = (\phi : 1) : C \to \mathbb{P}^1$$

which is *finite* since C is projective. For each $(y : 1) \in \mathbb{P}^1$, the fiber:

$$\Phi^{-1}(y) = \{x_1, \ldots, x_k\}$$

is a finite set, each of whose elements can be assigned a *ramification index* corresponding to the multiplicity of the rational function $\phi - y$ at x_i. Similarly, in the case of the point $(0 : 1)$, one defines the multiplicity of the function ϕ^{-1} at x_i. But the sum of these ramification indices over the set of x_i is the degree of the (finite) field extension $\mathbb{C}(\mathbb{P}^1) \subset \mathbb{C}(C)$, so these sums are all the same, and in particular $\deg(\operatorname{div}(\phi))$ is:

$$(\text{sum of ram indices over } (0:1)) - (\text{sum of ram indices over } (1:0))$$

which is zero.

Definition: Divisors D and D' on a smooth projective curve C are *linearly equivalent*, written $D \sim D'$, if:

$$D - D' = \operatorname{div}(\phi) \text{ for some } \phi \in \mathbb{C}(C)$$

Observations: (i) Linear equivalence is an equivalence relation.

(ii) Suppose D and D' are effective divisors. Then $D \sim D'$ if and only if there is a regular map $\Phi : C \to \mathbb{P}^1$ and points $y, y' \in \mathbb{P}^1$ such that $\Phi^{-1}(y) = D$ (counting ramification multiplicities) and $\Phi^{-1}(y') = D'$.

(iii) The degrees of linearly equivalent divisors are equal.

Let D be an effective divisor on a smooth, projective curve C.

Proposition 8.3: (a) The set:

$$L(D) := \{0\} \cup \{\phi \in \mathbb{C}(C) \mid \operatorname{div}(\phi) + D \text{ is effective}\}$$

is a sub-vector space of $\mathbb{C}(C)$ of dimension $\leq \deg(D) + 1$.

(b) The set of effective divisors linearly equivalent to D:

$$|D| := \{D' \mid D' \sim D\}$$

"is" the projective space of lines through the origin in $L(D)$.

Proof: The multiplicity at x of the sum of two rational functions is bounded above by the maximum of the two multiplicities, since:

$$\phi = u_x\pi_x^d, \psi = v_x\pi_x^e \;\Rightarrow (\phi+\psi) = (u_x + v_x\pi_x^{e-d})\pi_x^e$$

(assuming $d \le e$). Thus it follows immediately that if $\mathrm{div}(\phi) + D$ and $\mathrm{div}(\psi) + D$ are effective, then so too is $\mathrm{div}(\phi+\psi) + D$, which gives the linear structure to $L(D)$. As for the dimension, we proceed by induction, starting with the "zero" divisor 0:

$$L(0) = \mathbb{C}$$

since only the constant functions are regular on C. Next,

$$\dim(L(D+p)) \le \dim(L(D)) + 1$$

for the following reason. Suppose $\phi, \psi \in L(D+p) - L(D)$. Then in particular, ϕ, ψ have the *same* multiplicity at p, so:

$$\phi = u_x\pi_x^d \text{ and } \psi = v_x\pi_x^d$$

for units $u_x, v_x \in \mathcal{O}_x$. Therefore, there is a scalar $c = u_x(x)/v_x(x) \in \mathbb{C}$, such that $\phi - c\psi \in L(D)$, as desired.

Definition: $l(D) = \dim(L(D))$ and $r(D) = \dim(|D|) = l(D) - 1$.

Riemann-Roch Question: Given a smooth projective curve C and an effective divisor $D = \sum d_i x_i$ on C, Proposition 8.3 tells us:

$$r(D) \le \deg(D)$$

But can we get more precise information about the value of $r(D)$?

Abel Question: There is a smooth, projective variety:

$$C_d := C \times \cdots \times C/S_d \text{ (the quotient by the symmetric group)}$$

parametrizing the effective divisors of degree d. This variety is a union of the projective spaces $|D|$, which are the linear equivalence classes of effective divisors. Are these projective spaces the fibers of a regular map $\Phi : C_d \to X_d$? If so, what is X_d?

Trivial Example: Suppose $C = \mathbb{P}^1$ and $D = \sum d_i x_i$. Then there is a unique (up to scalar multiple) homogeneous polynomial $G \in \mathbb{C}[x,y]_d$ with the property that the roots of G (counted with multiplicity) precisely sum to the divisor D. It then follows that:

$$L(D) = \{0\} \cup \left\{ \phi = \frac{F}{G} \mid F \in \mathbb{C}[x,y]_d \right\} \text{and so } l(D) = d+1$$

attains the maximum value. Moreover, it follows that:

$$|D| = \mathbb{P}^d = \mathbb{P}^1_d$$

accounts for **all** the effective divisors of degree d, so X_d is a point.