2. Abstract Varieties. An abstract variety is a set with a (Zariski) topology and a sheaf of functions that is locally affine and separated. In order to define these terms properly, we need to define them in the context of an appropriate category.

Let $X$ be a topological space and $A$ be a commutative ring with 1.

**Definition.** A sheaf $\mathcal{S}$ of $A$-valued functions on $X$ consists of:

(a) A commutative ring with 1, denoted $\mathcal{S}(U)$, consisting of a subring of the ring of functions $f : U \to A$ for each open $U \subset X$, such that:

(b) functions in $\mathcal{S}(V)$ restrict to functions in $\mathcal{S}(U)$ if $U \subseteq V$, and

(c) functions $f_i \in \mathcal{S}(U_i)$ that agree when restricted to intersections $U_i \cap U_j$ are restrictions of a single well-defined function $f \in \mathcal{S}(\bigcup U_i)$.

**Examples:** (a) The sheaf of “totally discontinuous” functions on $X$. In this sheaf, denoted $A_X^{\text{disc}}$, each $A_X^{\text{disc}}(U)$ consists of all $f : U \to A$.

(b) At the other extreme, the constant functions do not form a sheaf. But there is a sheaf, denoted $A_X$, consisting of functions that are locally constant. In this sheaf, $A_X(U)$ consists of all the functions $f : U \to A$ that are constant on each connected component of $U$.

(c) If $A$ has a topology (e.g. $\mathbb{R}$ or $\mathbb{C}$ with the Euclidean topology), then the functions $f : U \to A$ that are continuous form a sheaf, often denoted simply by $\mathcal{C}_X$. This generalizes examples (a) and (b) (Why?)

(d) If $X$ is a differentiable manifold, then the rings of “infinitely differentiable” functions $f : U \to \mathbb{R}$ form a sheaf, denoted by $\mathcal{C}_X^\infty$.

(e) The regular functions on an affine variety $X$ with the Zariski topology form a sheaf of $\mathbb{C}$-valued functions, denoted $\mathcal{O}_X$, which has the unusual property that $U \subseteq V$ implies $\mathcal{O}_X(V) \subseteq \mathcal{O}_X(U)$. In other words, the restriction of such functions is an injective map for this sheaf. This is not true of any of the other sheaves discussed here (Why?)

(f) Suppose $V \subset X$ is an open subset and $\mathcal{S}$ is a sheaf of $A$-valued functions on $X$. Then the induced topology on $V$, together with the “restricted sheaf” $\mathcal{S}|_V$ defined by: $\mathcal{S}|_V(U) := \mathcal{S}(U)$ for all $U \subseteq V$ is a sheaf of $A$-valued functions on $V$. 
Definition: A morphism between pairs \((X, \mathcal{S}_X)\) and \((Y, \mathcal{S}_Y)\) consisting of a topological space with a sheaf of \(A\)-valued functions is:

(a) A continuous map \(F : X \to Y\) with the property that:
(b) the pull-back on functions defined by \(F^*(f) := f \circ F\) maps each:

\[
F^* : \mathcal{S}_Y(U) \to \mathcal{S}_X(F^{-1}(U))
\]

This gives the collection of pairs \((X, \mathcal{S}_X)\) the structure of a category.

Examples: (a) For any \(X\), the identity map defines a morphism:

\[
id : (X, A_X^{\text{disc}}) \to (X, A_X)
\]

but not in the opposite direction (unless \(X\) is discrete)! Similarly,

(b) The identity map on a differentiable manifold \(X\) defines:

\[
id : (X, \mathcal{C}_X) \to (X, \mathcal{C}_X^\infty)
\]

but not in the opposite direction.

(c) If \(t : V \subset X\) is an open set together with the restricted sheaf \(\mathcal{S}|_V\) (for any sheaf \(\mathcal{S}\) of \(A\)-valued functions), then the inclusion map

\[
t : (V, \mathcal{S}|_V) \to (X, \mathcal{S})
\]

is a morphism.

(d) A continuous mapping of differentiable manifolds \(F : X \to Y\) is (infinitely) differentiable if and only if it defines a morphism:

\[
F : (X, \mathcal{C}_X^\infty) \to (Y, \mathcal{C}_Y^\infty)
\]

i.e. it pulls back (locally) infinitely differentiable functions on \(Y\) to (locally) infinitely differentiable functions on \(X\).

(e) Let \(X \subset \mathbb{C}^n\) and \(Y \subset \mathbb{C}^m\) be irreducible algebraic sets, with Zariski topologies and coordinate rings \(\mathbb{C}[X]\) and \(\mathbb{C}[Y]\). A morphism \(F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)\) in the category of topological spaces with sheaves of \(\mathbb{C}\)-valued functions determines a \(\mathbb{C}\)-algebra homomorphism \(F^* : \mathcal{O}_Y(Y) = \mathbb{C}[Y] \to \mathbb{C}[X] = \mathcal{O}_X(X)\), and conversely:

Proposition 2.1: Each \(\mathbb{C}\)-algebra homomorphism \(\Phi : \mathbb{C}[Y] \to \mathbb{C}[X]\) comes from a uniquely determined morphism:

\[
F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)
\]

in the sense that \(\Phi = F^* : \mathbb{C}[Y] \to \mathbb{C}[X]\).
**Proof:** Recall that the points of $X$ are in a natural bijection with the maximal ideals of $\mathbb{C}[X]$. Thus:

$$F(x) = y \iff \Phi^{-1}(m_x) = m_y$$

is well-defined, and it is the only possible map for which $F^* = \Phi$ on (globally defined) regular functions. Moreover,

- $F^{-1}(U_g) = U_{\Phi(g)}$ for all $g \in \mathbb{C}[Y]$, so $F$ is continuous, and
- $F^*(f/g^n) = \Phi(f)/\Phi(g)^n$ shows that $F^* : O_Y(U_g) \to O_X(U_{\Phi(g)})$

from which it follows that $F$ is a morphism, as desired.

**Corollary 2.2:** Irreducible algebraic sets, with rational functions:

$$(X, O_X)$$ and $$(Y, O_Y)$$

are isomorphic in the category of topological spaces with sheaves of $\mathbb{C}$-valued functions if and only if $\mathbb{C}[X] \cong \mathbb{C}[Y]$ as $\mathbb{C}$-algebras.

**Refined Definition (of an affine variety):** A complex affine variety is a topological space $X$ with sheaf $S_X$ of $\mathbb{C}$-valued functions that is isomorphic to some $(Y, O_Y)$, where $Y$ is an irreducible algebraic set in some $\mathbb{C}^m$ and $O_Y$ is its sheaf of regular functions.

**Important Example:** If $X = V(P) \subset \mathbb{C}^n$ and $f \in \mathbb{C}[x_1, \ldots, x_n]$, then $(U_f, O_X|_{U_f})$ (for $U_f \subset X$) is an affine variety. It is isomorphic to the “affine hyperbola over $U_f$,” namely

$$Y := V(P, 1 - fx_{n+1}) \subset \mathbb{C}^{n+1}$$

which is an irreducible algebraic set with coordinate ring:

$$\mathbb{C}[Y] \cong \mathbb{C}[X][x_{n+1}]/(1 - fx_{n+1}) \cong \mathbb{C}[X][f^{-1}] \subset \mathbb{C}(X)$$

**Definition:** A pair $(X, S_X)$ consisting of a topological space with sheaf of $\mathbb{C}$-algebras is a prevariety if $X$ is connected and covered by open sets:

$$X = \bigcup_{i=1}^n U_i$$

with the property that each of the pairs $(U_i, S_X|_{U_i})$ is an affine variety.

**Corollary (of the important example) 2.3:** Every (Zariski) open subset $U \subset X$ (with sheaf $O_X|_U$) of an affine variety is a prevariety.

**Exercise 2.1:** The open subset $U = \mathbb{C}^2 - \{(0,0)\} \subset \mathbb{C}^2$ with sheaf $O_U = O_{\mathbb{C}^2}|_U$ is a prevariety but not an affine variety.

**Gluing:** Let $(X, S_X)$ and $(Y, S_Y)$ be topological spaces with sheaves of $A$-valued functions that have isomorphic open subsets, specifically $U \subset X$ and $V \subset Y$ with an isomorphism $F : (U, S_X|_U) \cong (V, S_Y|_V)$. Then we may “glue $X$ and $Y$ along $F$” to obtain $(Z, S_Z)$ defined by:
As a set,
\[ Z = (X \coprod Y) / \sim \text{ where } x \sim f(x) \text{ for each } x \in U \]
with natural inclusion maps \( \iota_X : X \subset Z \) and \( \iota_Y : Y \subset Z \).

- \( W \subset Z \) is open if and only if both \( W \cap X \) and \( W \cap Y \) are open.
- \( S_Z(W) \) is the set of functions \( f : W \to A \) satisfying:
  \[ f|_{W \cap X} \in S_X(W \cap X), \ f|_{W \cap Y} \in S_Y(W \cap Y) \]
and \( F^*(f|_{W \cap V}) = f|_{W \cap U} \).

**Conversely:** Given \( (X, S_X) \) and open sets \( U \subset X \) and \( V \subset X \), then \( (X, S_X) \) is isomorphic to the topological space with sheaf of \( A \)-valued functions obtained by gluing \( (U, S_X|_U) \) to \( (V, S_X|_V) \) along the canonical isomorphism \( F : U \cap V \cong V \cap U \).

**Corollary 2.3:** A prevariety is obtained by gluing an affine variety to another affine variety (or prevariety) along non-empty open subsets.

**Two Very Different Examples:** One can glue
\[ (\mathbb{C}, \mathcal{O}_\mathbb{C}) \text{ to } (\mathbb{C}, \mathcal{O}_\mathbb{C}) \]
along \( \mathbb{C}^* = \mathbb{C} - \{0\} \) in two different ways:

(i) Gluing along the identity isomorphism \( id : \mathbb{C}^* \to \mathbb{C}^* \) produces “the affine line with doubled origin.” In the framework of manifolds, this is a simple example of a non-Hausdorff “fake” manifold. Unfortunately, the Zariski topology on a variety is essentially never Hausdorff (since all pairs of nonempty open subsets of an affine variety intersect). Thus, we will have to find another way to eliminate it.

However, there is another interesting automorphism of \( \mathbb{C}^* \). Since:
\[ (\mathbb{C}^*, \mathcal{O}_{\mathbb{C}^*}) \cong (X, \mathcal{O}_X) \]
where \( \mathbb{C}[X] \cong \mathbb{C}[x, x^{-1}] \) by the affine hyperbola construction, it follows that the \( \mathbb{C} \)-algebra automorphism:
\[ \mathbb{C}[X] \to \mathbb{C}[X]; \ x \mapsto x^{-1} \]
is associated to an automorphism \( F \) of \( (\mathbb{C}^*, \mathcal{O}_{\mathbb{C}^*}) \).

(ii) Gluing along \( x \mapsto x^{-1} \) produces the projective line.

(We will explore this in detail later.)

**Definition:** A *product* of objects \( X, Y \) of a category \( \mathcal{C} \) is an object, which we will denote by \( X \times Y \), together with “projection” morphisms \( \pi_X : X \times Y \to X \) and \( \pi_Y : X \times Y \to Y \) with the following:
Universal property: To each triple \((Z; F, G)\) consisting of an object \(Z\), with morphisms \(F : Z \to X\) and \(G : Z \to Y\), there is a unique:
\[ Z \to X \times Y \]
that commutes with the morphisms to \(X\) and \(Y\).

The product is unique (if it exists) up to a uniquely determined isomorphism. Thus we can get away the leading notation “\(X \times Y\).”

Example: The Cartesian product is a product in the category of sets. In the category of topological spaces, the Cartesian product, together with the product topology, is a product.

Lookup: The tensor product \(C[X] \otimes C[Y]\) of integral domains that are finitely generated as \(C\)-algebras is also an integral domain, and also finitely generated as a \(C\)-algebra.

Corollary 2.4: If \(X\) and \(Y\) are affine varieties, then
\[ C[X \otimes C[Y] \cong C[X \times Y] \]
for an affine variety \(X \times Y\), which, together with morphisms \(\pi_X\) and \(\pi_Y\) associated to the inclusions \(C[X] \subset C[X \times Y]\) and \(C[Y] \subset C[X \times Y]\), respectively, is the product of \(X\) and \(Y\) in the category of affine varieties. This is due to the analogous universal property of the tensor product.

Warning: Note that \(C^m \times C^n \cong C^{m+n}\), but that, as we’ve already noted in an earlier exercise, the Zariski topology on this product is not in general equal to the product topology. This does not contradict the Example above describing the products of topological spaces (why?).

Definition: In a category whose objects are topological spaces, whose morphisms are continuous, and in which products exist, an object \(X\) is separated if the image of the canonical diagonal map
\[ \delta : X \to X \times X \]
is closed.

Exercise 2.2: (a) In the category of topological spaces, prove that \(X\) is Hausdorff if and only if it is separated.

(b) Prove that as a set, the product \(X \times Y\) of affine varieties is the Cartesian product of \(X\) and \(Y\) (this is not, however, true of schemes).

Proposition 2.5: Affine varieties are separated.

Proof: Let \(X\) be an affine variety. Via an isomorphism we may assume \(X = V(P) \subset C^n\) with coordinates \(x_1, \ldots, x_n\). Then \(X \times X \subset C^{2n}\) with coordinates \(x_1, \ldots, x_n, y_1, \ldots, y_n\), and \(\delta(X) \subset X \times X\) is the closed subset defined by the equations \(\{x_i - y_i = 0 \mid i = 1, \ldots, n\}\).

More Involved Exercises 2.3:
(a) Prove that products exist in the category of prevarieties.

(b) A quasi-affine variety is, by definition, a pair $(U, \mathcal{O}_X|_U)$, where $U \subset X$ is a (non-empty) open subset of an affine variety $X$. Prove that quasi-affine varieties are separated.

(c) Prove that the affine line with the doubled origin is not separated.

(d) Prove that the projective line is separated.

Definition: An abstract variety is a separated prevariety.