

# Complex Algebraic Geometry: Smooth Curves

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**11. The Riemann-Roch Theorem.** The Riemann-Roch theorem gives a precise formula for the *difference*  $l(D) - l(K_C - D)$  when  $D$  is a divisor on a smooth, projective curve  $C$ . This leads to the introduction of a third definition of the genus of  $C$  (equivalent to the other two), and gives an enormous amount of information about the maps and closed embeddings of  $C$  into projective space.

**Riemann-Roch Theorem:** If  $C \subset \mathbb{P}^n$  is a smooth, projective curve of (arithmetic or topological) genus  $g$ , then for all divisors  $D$  on  $C$ :

$$l(D) - l(K_C - D) = \deg(D) + 1 - g$$

**Proof:** Let  $d$  be the degree of  $C \subset \mathbb{P}^n$ , and consider:

$$D' = \text{div}(G), \text{ where } 0 \neq G \in \mathbb{C}[C]_t \text{ and } t \gg 0$$

Then:

- (a)  $\mathbb{C}[C]_t \cong L(D')$  (by Proposition 10.1(a))
- (b)  $\deg(D') = dt$ , and
- (c)  $\deg(K_C - D') = (2g - 2) - dt < 0$ , so  $l(K_C - D') = 0$ . Thus:

$$l(D') - l(K_C - D') = H_C(t) = dt + 1 - g$$

Suppose now that  $D = \sum_i d_i x_i$  is an arbitrary divisor. We can always find a divisor of the form  $D'$  as above so that:

$$D' = \sum_i d'_i x_i, \text{ and each } d'_i \geq d_i.$$

Let  $\pi_i$  be a uniformizing parameter at  $x_i$  and consider the vector space:

$$V := \left\{ \sum_i a_{i,d'_i} \pi_i^{-d'_i} + \cdots + a_{i,d_i+1} \pi_i^{-d_i+1}; a_{i,j} \in \mathbb{C} \right\}$$

Then there is a natural map  $e : L(D') \rightarrow V$ , defined by:

$$\phi \mapsto \{ a_{i,d'_i} = (\phi \pi_i^{d'_i})(x_i), a_{i,d'_i-1} = \left( (\phi - a_{i,d'_i} \pi_i^{-d'_i}) \pi_i^{d'_i-1} \right) (x_i), \cdots \}$$

capturing the portion of the “Laurent series” at each point  $x_i$  between  $-D$  and  $-D'$ . In particular, the *kernel* of  $e$  is precisely  $L(D) \subset L(D')$ , the rational functions whose poles do not register in  $V$ . That is,

$$0 \rightarrow L(D) \rightarrow L(D') \rightarrow V$$

is left exact, from which it immediately follows that:

$$\deg(D') - \deg(D) = \dim(V) \geq l(D') - l(D)$$

and, using Riemann-Roch for  $D'$ ,

$$l(D) \geq l(D') - \deg(D') + \deg(D) = \deg(D) + 1 - g$$

which is the Riemann-Roch *Inequality*.

To get equality, we will need to use some properties of differentials. Suppose  $K_C = \text{div}(\omega_0)$  is a canonical divisor, associated to a particular differential form  $\omega_0$ . Then:

$$L(K_C - D) = \{0\} \cup \{\phi \in \mathbb{C}(C) \mid \text{div}(\phi) + K_C \geq D\}$$

(where  $D \geq E$  if  $D - E$  is effective), and, since  $\text{div}(\phi) + K_C = \text{div}(\phi\omega_0)$ ,

$$L(K_C - D) = \{0\} \cup \{\omega = \phi\omega_0 \in \Omega(C) \mid \text{div}(\omega) \geq D\}$$

The main idea is to pair differentials with elements of  $V$  by means of the *residue*. This is a unique property of differentials, which is perhaps best expressed in terms of one complex variable analysis. If  $\omega \in \Omega(C)$ , then in particular,  $\omega$  is a meromorphic differential form, which can be integrated along small loops  $\gamma$  winding once around a point  $x \in C$ . Then the *residue* of  $\omega$  at  $x$  is:

$$\text{Res}_x(\omega) := \frac{1}{2\pi i} \int_{\gamma} \omega$$

In terms of a uniformizing parameter  $\pi_x$ , if we write:

$$\omega = (a_{-d}\pi_x^{-d} + \cdots + a_{-1}\pi_x^{-1}) d\pi_x + \tau$$

where  $\tau \in \Omega_x$  is regular at  $x$ , then:

$$\text{Res}_x(\omega) = a_{-1}$$

showing that this coefficient does not depend upon the choice of  $\pi_x$ . Moreover, from the analysis, we also get:

**Stoke's Theorem:** For  $\omega \in \Omega(C)$ , the sum:

$$\sum_{x \in C} \text{Res}_x(\omega) = 0$$

Consider the pairing:  $L(K_C - D) \times V \rightarrow \mathbb{C}$ ;  $(\omega, \psi) \mapsto \sum_i \text{Res}_{x_i}(\psi\omega)$ . By construction of  $V$ , the kernel of the induced map to the dual space:

$$f : L(K_C - D) \rightarrow V^\vee$$

is  $L(K_C - D')$  (every differential form with zero divisor between  $D$  and  $D'$  can be paired with a suitable Laurent tail so that  $a_{-1} \neq 0$  for precisely one of the  $x_i$ ). But  $L(K_C - D) = 0$ , so  $f$  is injective.

Moreover, when paired with elements  $\psi \in L(D') \subset V$ , we obtain:

$$(\omega, \psi) \mapsto \sum \text{Res}_{x_i}(\psi\omega) = 0$$

from Stoke's Theorem, since here the  $x_i \in C$  are the *only* poles of  $\psi\omega$ . Thus, we have the long exact sequence of vector spaces:

$$0 \rightarrow \ker \rightarrow V^\vee \rightarrow L(D')^\vee \rightarrow L(D)^\vee \rightarrow 0$$

with  $L(K_C - D) \subset \ker$ . Now we compute:

$$l(K_C - D) \leq \dim(V) - l(D') + l(D) = l(D) - \deg(D) - 1 + g$$

and so,

$$l(D) - l(K_C - D) \geq \deg(D) + 1 - g$$

Reversing the roles of the divisors  $D$  and  $K_C - D$  gives the opposite inequality, hence the Theorem.

This brings us to the final definition of the genus:

**Definition.** The *geometric genus* of a smooth projective curve  $C$  is:

$$g_{\text{geom}}(C) = l(K_C) = \dim \Omega[C]$$

**Corollary (of Riemann-Roch):**

$$g_{\text{top}}(C) = g_{\text{arith}}(C) = g_{\text{geom}}(C)$$

**Proof:** Plug  $D = 0$  into the Riemann-Roch Theorem:

$$1 - g_{\text{geo}}(C) = l(0) - l(K_C) = 0 + 1 - g_{\text{top}}(C)$$