

# Complex Algebraic Geometry: Smooth Curves

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**10. The Genus of a Curve.** We will introduce two notions of the genus of a smooth projective curve, and prove that they are the same. This will require the Riemann-Hurwitz Theorem for maps of smooth curves, and some considerable analysis of the properties of the projections of smooth curves into  $\mathbb{P}^2$  and onto  $\mathbb{P}^1$ .

**Definition:** The *canonical divisor* (class) on a smooth projective curve  $C$  is any divisor (preferably, but not necessarily, effective) satisfying:

$$K_C = \text{div}(\omega)$$

for some rational differential form  $\omega$ . Notice that, in fact,  $K_C$  is effective if and only if  $\omega \in \Omega[C]$ .

**Topological Genus:** The degree of a canonical divisor:

$$\deg(K_C) = \deg(\text{div}(\omega))$$

is used to define the *topological genus*  $g_{\text{top}}(C)$  of  $C$  by the formula:

$$\deg(K_C) = 2g_{\text{top}} - 2$$

Our first task is to see that what this has to do with topology.

Suppose  $\Phi : C \rightarrow B$  is a finite mapping of smooth projective curves. Recall that this induces a finite extension of fields:

$\mathbb{C}(B) \subset \mathbb{C}(C)$  of degree equal to the degree of the map  $\Phi$

For each point  $x \in C$  mapping to  $y = \Phi(x) \in B$ , we define the *ramification index*  $e_\Phi(x)$  of  $\Phi$  at  $x$  to be the multiplicity of  $\pi_y$  at  $x$  (thought of as an element of  $\mathbb{C}(C)$ ). This allows us to define the inverse image  $\Phi^{-1}(y)$  as an effective divisor of degree  $d = \text{degree of } \Phi$ :

$$D_y := \sum_{\Phi(x)=y} e_\Phi(x)x$$

It also allows us to define the *ramification divisor* for the map by:

$$R_\Phi := \sum_{x \in C} (e_\Phi(x) - 1)x$$

(this is an effective divisor since every point has ramification index  $\geq 1$ , and only finitely many points have ramification index  $> 1$ ).

**Riemann-Hurwitz Theorem:** A canonical divisor on  $C$  is obtained from a canonical divisor  $K_B = \sum n_i y_i$  on  $B$  via:

$$K_C := \sum_1 n_i D_{y_i} + R$$

**Proof:** The canonical divisor  $K_B$  is  $\text{div}(\omega)$  for some differential form  $\omega = \phi d\psi$ . We consider the same expression, but instead as a differential form on  $C$ . The result now follows immediately from the fact that:

$$d(\pi_x^e) = ex^{e-1}d\pi_x$$

for a uniformizing parameter  $\pi_x$  at  $x \in C$ .

**Corollary 1:** The degrees of canonical divisors are related via:

$$\deg(K_C) = (\deg(\Phi)) \deg(K_B) + \deg(R)$$

**Corollary 2:** The topological genus of  $C$  is the number of holes, when we regard  $C$  as an orientable (real) closed surface.

**Proof:** The example in §9 gave the (rational) differential form:

$$dx = -\frac{dy}{y^2}$$

on  $\mathbb{P}^1$ , which gives a canonical divisor of degree  $-2$ . Thus  $g_{\text{top}}(\mathbb{P}^1) = 0$ , and, of course,  $\mathbb{P}^1$  is the Riemann *sphere*. Next, suppose  $C$  is a smooth projective curve, and consider the map:

$$\Phi : C \rightarrow \mathbb{P}^1$$

determined by the choice of a non-constant  $\phi \in \mathbb{C}(C)$  (as in §8). Recall that the number of holes of  $C$  is given by the formula:

$$(2 - 2(\#\text{holes})) = \#V - \#E + \#F$$

where  $V, E, F$  are the vertices, edges and faces of a *triangulation* of  $C$ . But now let  $S \subset \mathbb{P}^1$  be the image of the (finite) set of points of  $x \in C$  with  $e_\Phi(x) > 1$ , and consider a triangulation of the Riemann sphere  $V_{\mathbb{P}^1}, E_{\mathbb{P}^1}, F_{\mathbb{P}^1}$  with the property that  $S \subset V$ , which is fine enough so that  $\Phi^{-1}(V), \Phi^{-1}(E), \Phi^{-1}(F)$  is a triangulation of  $C$ . In this triangulation, there are  $dE_{\mathbb{P}^1}$  edges,  $dF_{\mathbb{P}^1}$  faces, and  $dV_{\mathbb{P}^1} - \deg(R)$  vertices(!), and it therefore follows that  $g_{\text{top}}$  is the number of holes.

**Arithmetic Genus:** The *arithmetic genus*  $g_{\text{arith}}(C)$  of an *embedded* smooth, projective curve  $C \subset \mathbb{P}^n$  is defined by the formula:

$$H_C(t) = dt + 1 - g_{\text{arith}}(C)$$

where  $H_C(t)$  is the Hilbert polynomial of  $C$ , and  $d$  is the degree of  $C$ .

**Example:** Suppose  $C \subset \mathbb{P}^2$  is smooth, defined by a homogeneous polynomial  $F(x, y, z)$  of degree  $d$ . Then the ideal  $I(C)$  decomposes as:

$$I(C)_t = \mathbb{C}[x, y, z]_{t-d} \cdot F \subset \mathbb{C}[x, y, z]_t \text{ for } t \gg 0$$

and so:

$$\begin{aligned} H_C(t) &= \dim(\mathbb{C}[x, y, z]_t) - \dim(\mathbb{C}[x, y, z]_{t-d}) = \binom{t+2}{2} - \binom{t-d+2}{2} \\ &= \frac{(t^2 + 3t + 2) - (t^2 + 3t + 2 - 2dt + d^2 - 3d)}{2} = dt + 1 - \binom{d-1}{2} \end{aligned}$$

In other words, the arithmetic genus satisfies:

$$p_{\text{arith}}(C) = \binom{d-1}{2}$$

(and note that not every genus occurs).

On the other hand, consider the *projection* from  $p = (a : b : c) \notin C$ :

$$\pi_p : C \rightarrow \mathbb{P}^1$$

This is a regular map of degree  $d$  that we can use to compute the *topological* genus of  $C$ , provided that we can determine the ramification. But the tangent line to  $C$  at  $p'$  passes through  $p$  if and only if:

$$a \cdot \frac{\partial F}{\partial x}(p') + b \cdot \frac{\partial F}{\partial y}(p') + c \cdot \frac{\partial F}{\partial z}(p') = 0$$

and as a result, the points on  $C = V(F)$  that are ramification points for the projection from  $p'$  are all the points of:

$$V(F) \cap V\left(a \cdot \frac{\partial F}{\partial x} + b \cdot \frac{\partial F}{\partial y} + c \cdot \frac{\partial F}{\partial z}\right),$$

the intersection of curves of degrees  $d$  and  $d-1$ . This consists of:

$$d(d-1) \text{ points}$$

of minimal ramification ( $e = 2$ ) if  $p$  is chosen “generically,” and then:

$$\deg(K_C) = -2d + d(d-1) = d^2 - 3d = 2 \binom{d-1}{2} - 2$$

so that in this example:

$$g_{\text{arith}}(C) = \binom{d-1}{2} = g_{\text{top}}(C)$$

In the general case, we need project our curves to  $\mathbb{P}^2$ . Suppose:

$$C \subset \mathbb{P}^2 \text{ is a smooth curve of degree } d$$

We will take the following as a given:

(\*) A “generic” projection  $\pi : C \rightarrow \mathbb{P}^2$  has image equal to a curve:

$$B \subset \mathbb{P}^2$$

of degree  $d$  with  $\nu$  simple nodes as singularities.

The really key idea is the following important:

**Proposition 10.1:** (a) Suppose  $C \subset \mathbb{P}^n$  is a smooth projective curve that is not contained in any hyperplane, and suppose  $0 \neq G \in \mathbb{C}[C]_t$ . Let  $D = \text{div}(G)$  (on the curve) and consider the map:

$$r_t : \mathbb{C}[C]_t \rightarrow L(D); \quad F \mapsto \text{div} \left( \frac{F}{G} \right)$$

Then  $r_t$  is injective for all  $t$ , and surjective for  $t \gg 0$ . In particular, the Hilbert polynomials of  $C$  and of a projection,  $\pi(C) \subset \mathbb{P}^m$ , are **the same** if  $C$  is isomorphic to  $\pi(C)$ .

(b) In case (\*) above, the Hilbert polynomials of  $C$  and of  $B$  satisfy:

$$H_t(C) = H_t(B) + \nu$$

We will prove a more general version of the Proposition later. With it, we may follow the Example above to show:

$$g_{\text{arith}}(C) = g_{\text{top}}(C)$$

**Step 1:** As in the Example,

$$H_B(t) = dt + 1 - \binom{d-1}{2} \Rightarrow H_C(t) = dt + 1 - \left( \binom{d-1}{2} - \nu \right)$$

so

$$g_{\text{arith}}(C) = \binom{d-1}{2} - \nu$$

**Step 2:** Consider a generic projection:

$$\pi_p : B = V(F) \rightarrow \mathbb{P}^2; \quad p = (a : b : c)$$

As in the Example, the ramification points will be points of:

$$V(F) \cap V \left( a \cdot \frac{\partial F}{\partial x} + b \cdot \frac{\partial F}{\partial y} + c \cdot \frac{\partial F}{\partial z} \right)$$

However, in this case, each of the  $\nu$  nodes of  $B$  also belong to this intersection, and, by a local coordinate computation, *each node counts with multiplicity two*. Thus the number of ramification points is:

$$d(d-1) - 2\nu$$

and by Riemann-Hurwitz, applied to the map  $\pi : C \rightarrow B \rightarrow \mathbb{P}^1$ ,

$$\deg(K_C) = -2d + d(d-1) - 2\nu = 2 \left( \binom{d-1}{2} - \nu \right) - 2$$

and so,

$$g_{\text{top}}(C) = \binom{d-1}{2} - \nu = g_{\text{arith}}(C)$$