Transformational Geometry of the Plane (Master Plan)

Day 1. Some Coordinate Geometry.

- Cartesian (rectangular) coordinates on the plane.
- What is a line segment?
- What is a (right) triangle?
- State and prove the Pythagorean Theorem.
- Find the distance between two points.
- What is a line?
- What is the slope of a line?
- When are two lines parallel? Perpendicular?
- Recall the "different" equations for a line in the plane.
- Find the intersection point of two lines from their equations.
- What are the coordinates of the midpoint of a line segment?
- What is a median of a triangle?
- Show that the medians of a triangle meet in a point (find it!).
- What is an altitude of a triangle?
- Show that the altitudes of a triangle meet in a point (find it!).

Day 2. Vectors.

- What is a vector (and how does it differ from a point)?
- What is the vector difference between two points?
- What is the length of the vector A? (Call it $|A|$)
- What does multiplying a vector by a real number do to it?
- What is the unit vector in the direction of the Vector A?
- Why does every unit vector have the form:

  $$u = (\cos(\theta), \sin(\theta))$$

  for some angle θ?
- How do we add a vector to a point?
- What is the sum of two vectors?
- How is a vector "like" the slope of a line?
- Describe the line through P (point) in the direction A (vector).
- Intersect two lines described in this way.
- Find an equation for a line described in this way.
- The dot ($\cdot$) symmetric and det ($\wedge$) products of vectors.
- What are $A \cdot A$ and $A \wedge A$?
- What do $A \cdot B = 0$ and $A \wedge B = 0$ mean geometrically?
- Find a rule for the signs of $A \cdot B$ and $A \wedge B$.
- Compute the following for $A = (a_1, a_2)$ and $B = (b_1, b_2)$

  $$(A \cdot B)^2 + (A \wedge B)^2$$

  and factor the result.

Day 3. Angles.

- Why do the angles of a triangle always sum to 180 degrees?
- How is the angle θ from Vector A to Vector B measured?
- Given Vectors A and B, then there is one Vector C such that:
  (a) C is a (real) multiple of A. (b) $C \cdot (B - C) = 0$.
  This is the **projection** of B on A. Draw pictures.
- Find a formula for C, given $A = (a_1, a_2)$ and $B = (b_1, b_2)$.
- Verify the formulas $|C| = |B\cos(\theta)|$ and $|B - C| = |B\sin(\theta)|$ where θ is the angle from vector A to vector B.
- Use the two previous formulas to verify:

  $$A \cdot B = |A||B|\cos(\theta) \text{ and } A \wedge B = |A||B|\sin(\theta)$$

  (including the sign!)
- Conclude that $|A \wedge B|$ is twice the area of the triangle AB.
- Verify the law of sines.
- Show that the angle bisectors of a triangle meet in a point. (E.g. The law of sines leads to a nice proof)
- Verify the law of cosines from the formula above.

Day 4. Matrices.

- Add 2 x 2 matrices. Is this associative? Commutative?
- Multiply. Is this associative? Commutative? Distributive?
- What are the additive and multiplicative identity matrices?
- Perform the following four multiplications:

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}, \quad \begin{bmatrix} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{bmatrix} \cdot \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$$

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \cos(s) & \sin(s) \\ \sin(s) & -\cos(s) \end{bmatrix} \cdot \begin{bmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{bmatrix}$$

  (Use angle addition and subtraction formulas to simplify.)
- What is the determinant of a 2 x 2 matrix?
- What is the inverse of a 2 x 2 matrix (if it is invertible)?
- Find the inverses of the following matrices (assume $a, b \neq 0$):

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{bmatrix}, \begin{bmatrix} \cos(s) & \sin(s) \\ \sin(s) & -\cos(s) \end{bmatrix}$$

- Convert the following pair of equations into a matrix equation:

  $$a_1 x + b_1 y = c_1$$

  $$a_2 x + b_2 y = c_2$$

  and use the inverse matrix to solve for the point of intersection.

Day 5. Linear Transformations.

- Think of A and B as vectors written vertically, and consider

  $$F(x,y) = xA + yB = \begin{bmatrix} a_1x + b_1y \\ a_2x + b_2y \end{bmatrix} = \begin{bmatrix} a_1 & b_2 \\ a_2 & b_2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

  as a **map** from the plane to itself, and find:

  $$F(0,0), \quad F(1,0) \text{ and } F(0,1)$$

  Such maps are called **linear transformations** of the plane. The vectors are written vertically for convenience, as we will see. From this point of view, the matrix equation from yesterday is:

  $$F(x,y) = (c_1, c_2) \text{ (written horizontally)}$$

  In other words, solving this equation is finding $F^{-1}(c_1, c_2)$.

- A map from the plane to the plane is **linear** if:

  $F(rP) = rF(P)$ for all real numbers r and points P, and

  $F(P+Q) = F(P) + F(Q)$ for all points P and Q

  Show that a linear transformation is a linear map.
- Suppose F is a linear map, and $F(1,0) = A$ and $F(0,1) = B$. Then conclude that F is the map:

  $$F(x,y) = xA + yB$$

  That is, linear maps are the same as linear transformations!
- If F is a linear map, show that:
  (i) $F(0,0) = (0,0)$.
  (ii) $F(-P) = -F(P)$ for all points P.
  (iii) $F(Q-P) = F(Q) - F(P)$ for all points Q and P.
  Conclude from this that a linear map takes **vectors** to **vectors**!
- Show that rotations through the origin and reflections across lines through the origin are linear maps but that other rotations and reflections are not linear maps.
- What does the following linear transformation do to the plane?

  $$F(x,y) = x\begin{bmatrix} 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Find vectors A and B so that $F(x,y) = xA + yB$ for each of:

  (a)-(c) Reflection about the x-axis, y-axis and the origin.
  (d) Dilation by a factor a in all directions.
  (e) Stretch in the x-direction by a and the y-direction by b.

- For each angle a, describe the transformations:

$$F(x,y) = x\begin{bmatrix}\cos(a)\\ \sin(a)\end{bmatrix} + y\begin{bmatrix}-\sin(a)\\ \cos(a)\end{bmatrix}$$

$$F(x,y) = x\begin{bmatrix}\cos(2a)\\ \sin(2a)\end{bmatrix} + y\begin{bmatrix}\sin(2a)\\ -\cos(2a)\end{bmatrix}$$

- We want next to compose a pair of linear transformations. If

$$F(x,y) = x\begin{bmatrix}a_1\\ a_2\end{bmatrix} + y\begin{bmatrix}b_1\\ b_2\end{bmatrix} = \begin{bmatrix}a_1x+b_1y\\ a_2x+b_2y\end{bmatrix} \text{ and}$$

$$G(x,y) = x\begin{bmatrix}c_1\\ c_2\end{bmatrix} + y\begin{bmatrix}d_1\\ d_2\end{bmatrix} = \begin{bmatrix}c_1x+d_1y\\ c_2x+d_2y\end{bmatrix}$$

then what is $F(G(x,y))$?

- Compare what you got with the columns of the matrix product:

$$\begin{bmatrix} a_1 & b_1\\ a_2 & b_2\end{bmatrix} \cdot \begin{bmatrix} c_1 & d_1\\ c_2 & d_2\end{bmatrix}$$

- Deduce the angle addition formulas for sin and cos from the fact that the composition of rotation by the angle a and rotation by the angle b is rotation by the angle $a+b$.
- Let:

$$\begin{aligned}
\mathrm{rot}_a &= \text{rotation by the angle } a \text{ (counterclockwise)}\\
\mathrm{ref}_a &= \text{reflection across the line } \theta = a \text{ (in polar coordinates)}\\
\mathrm{dil}_r &= \text{dilation (stretch) by the factor } r \text{ (in all directions)}\\
\mathrm{str}_{s,t} &= \text{stretch by } s \text{ in the } x \text{ direction and } t \text{ in the } y \text{ direction}
\end{aligned}$$

Find the matrices for each of these linear maps.

- A final basic map is the *skew* transformation sk_b with matrix:

$$\begin{bmatrix} 1 & b\\ 0 & 1\end{bmatrix}$$

Describe what this transformation does to vectors in the plane.

- Fill in the following table:

F	G	$F \circ G$
rot_a	ref_b	
ref_b	rot_a	
ref_a	ref_b	
ref_b	ref_a	
$\mathrm{rot}_{\frac{\pi}{4}}$	$\mathrm{str}_{\frac{1}{2},2}$	
$\mathrm{str}_{\frac{1}{2},2}$	$\mathrm{rot}_{\frac{\pi}{4}}$	

In the first four cases, $F \circ G$ is a basic linear map. In the last two cases find A and B so that $F(G(x,y)) = xA + yB$.

Day 6. Area Factors for Linear Maps.

- Choose interesting vectors A and B and draw the image of the unit square under the linear map $F(x, y) = xA + yB$.
- Do this for the column vectors of each of the five basic linear maps with your choice of angles and stretch and skew factors, and also for $F \circ G$ for the last two lines of the table above.
- If the unit square is translated by a vector A, show that its image is translated by $F(A)$. If the unit square is dilated by r, show that the image dilates by r, too.
- Show that the **area** of the image of the unit square is $|A \wedge B|$.
- Conclude that the area of the image of **any** region is the area of the original region times $|A \wedge B|$. This is the "area factor" of the linear map $F(x, y) = xA + yB$.
- Explain geometrically why the area factor of $F \circ G$ is the product of the area factor of F with the area factor of G.
- Show (with algebra) that:
$$\det(M) \cdot \det(N) = \det(M \cdot N)$$
verifying the geometric explanation you gave above.
- Show that any linear map is a composition of three basic maps:
$$F(x, y) = \mathrm{rot}_a \circ \mathrm{str}_{s,t} \circ \mathrm{sk}_b(x, y)$$
for appropriates choice of a, s, t, b with $s \geq 0$. The idea is to take the image of the unit square, and return it to a unit square by rotating, stretching and skewing it. This is subtle!
- What is the area factor for the linear map:
$$F(x, y) = \mathrm{rot}_a \circ \mathrm{str}_{s,t} \circ \mathrm{sk}_b(x, y)$$
in terms of a, s, t, b? When is the determinant of the matrix associated with F **negative**? How is ref_a put in this form?
- The complex numbers may be viewed as linear transformations. If we think of $a + bi$ as the vector (a, b), then what is the matrix for the **linear map** given by:
$$F(x, y) = (a + bi) \cdot (x + yi) \quad ?$$
- Express this matrix as a dilation composed with a rotation. What is its area factor? Note that if we multiply two of these, the dilation factors multiply and the rotation angles add.
- Draw a picture with some detail in the first quadrant and then draw its image under examples of the five basic maps.

Day 7. Symmetry.

- A linear map is a **symmetry** if it has a **length** factor of 1, i.e.

  $$|F(A)| = |A| \text{ for } \textbf{all} \text{ vectors } A$$

  It is a **similarity** if it has a length factor r other than 1. Notice that the diation dil_r is a similarity but a stretch $\mathrm{str}_{s,t}$ does not have a length factor if $|s|$ and $|t|$ are different. Show that a skew map sk_b also does not have a length factor.
- Show that the composition of two symmetries is a symmetry.
- Show that a similarity preserves the **angles** between vectors. In other words, if the angle between A and B is θ, then the angle between $F(A)$ and $F(B)$ is also θ.
- Show that every symmetry is either a rotation or reflection.
- Suppose $S \subset \mathbb{R}^2$ is a subset of the plane and F is a symmetry. We say that F is a symmetry of S if $F(S) = S$. Let P_4 be the (rotated) square with vertices: $(1,0)$, $(0,1)$ $(-1,0), (0,-1)$. There are 8 symmetries of P_4 (counting id). Find them all.
- The set of symmetries of S is a group; they are closed under compositions and inverses. Make an 8×8 composition table for the group of symmetries of P_4.
- Do the same for P_3, the equilateral triangle with one vertex at $(1,0)$ centered at the origin (what are the other vertices?).
- The **dihedral group** D_{2n} is the group of symmetries of the regular n-gon P_n with a vertex at $(1,0)$ centered at the origin. How many elements does D_{2n} have? What do they look like?
- The **cyclic group** C_n is the subgroup of D_{2n} consisting of all *rotational symmetries* of the n-gon P_n. Show that the set of reflectional symmetries of P_n is **not** a group.
- A set of elements **generates** a group if everything in the group is a composition of those elements. The cyclic group C_n is generated by one element, namely $x = \mathrm{rot}_{\frac{2\pi}{n}}$ because all the rotational symmetries of P_n are powers of x. Show that this x together with $y = \mathrm{ref}_0$ (across the x-axis) generate D_{2n}. In fact, show that the following list of elements is D_{2n}:

  $$\mathrm{id}, x, x^2, ..., x^{n-1}, y, xy, x^2y, ..., x^{n-1}y$$

- An equation satisfied by the generators is a **relation**. E.g.

  $$x^n = \mathrm{id} \ \text{ and } \ y^2 = \mathrm{id}$$

  because rotating n times and reflecting twice give the identity. But there is one more relation in D_{2n}. Check that $yx = x^{-1}y$ and redo the composition tables for D_8 and D_6 with algebra.

Day 8. Affine Transformations.

- The **translation** by the vector $V = (v_1, v_2)$ is the map:

$$T_V(x, y) = (x + v_1, y + v_2)$$

  This is not linear because it translates the origin, but it does map vectors to vectors (by the identity map!) so it has a length factor of 1 and it preserves angles, i.e. it is a symmetry. Check:

$$T_V \circ T_W = T_{V+W}$$

  and conclude that translations commute with each other.
- An **affine map** is a linear map followed by a translation:

$$(T_V \circ F)(x, y) = xA + yB + V = \begin{bmatrix} a_1x + b_1y + v_1 \\ a_2x + b_2y + v_2 \end{bmatrix}$$

  Translations do not commute with linear maps. Instead:

$$(F \circ T_V)(x, y) = (T_{F(V)} \circ F)(x, y)$$

  so $F \circ T_V$ is another affine map (with a different translation).
- Show that:
  (a) $(T_V \circ F) \circ (T_W \circ G) = T_{V+F(W)} \circ (F \circ G)$ and
  (b) $T_{F^{-1}(-V)} \circ F^{-1}$ is the inverse of $T_V \circ F$.
  and conclude that the set of affine maps is a group.
- The map $T_P \circ F \circ T_{-P}$ translates P to 0, lets F do its thing and then translates back. Discuss the fact that:

$$T_P \circ \mathrm{rot}_a \circ T_{-P} = \mathrm{rot}_{P;a}$$

  is the rotation around P by the angle a, and that:

$$T_Q \circ \mathrm{ref}_b \circ T_{-Q} = \mathrm{ref}_{Q;b}$$

  is the reflection across the line through Q at angle b.
- Given P and F, solve for V so that:

$$T_V \circ F = T_P \circ F \circ T_{-P}$$

  and conclude that maps of this type are affine maps!
- Given V and $a \neq 0$, find the (unique!) point P so that:

$$\mathrm{rot}_{P;a} = T_V \circ \mathrm{rot}_a$$

  so composing rotation by a with a translation is **always** rotation by a around another point. What goes wrong when $a = 0$?
- Given angles a, b and points P, Q, solve for R so that:

$$\mathrm{rot}_{P;a} \circ \mathrm{rot}_{Q;b} = \mathrm{rot}_{R;a+b}$$

  (this is messy!)

Day 9. Reflections and Glides

- When a reflection across a line through Q is written in the form:
$$\mathrm{ref}_{Q;b} = T_W \circ \mathrm{ref}_b$$
as an affine map, then the translation vector is $W = Q - \mathrm{ref}_b(Q)$. Show that this is **perpendicular** to the line of reflection.
- Conversely, suppose W is perpendicular to the line of reflection (i.e. the angle of W is $\frac{\pi}{2} + b$). Show that $T_W \circ \mathrm{ref}_b = \mathrm{ref}_{\frac{1}{2}W;b}$

  Definition. A **glide reflection** is an affine map of the form
$$T_U \circ \mathrm{ref}_{Q,b}$$
where U is **parallel** to the line of reflection (i.e. its angle is b).

- Given V and b, write $V = U + W$ where W has angle $\frac{\pi}{2} + b$ and U has angle b and conclude that $T_V \circ \mathrm{ref}_b$ is a glide reflection!

  Taking Stock. Every *affine symmetry* is one of the following:
  (a) A translation T_V.
  (b) A rotation by angle a around a point P ($\mathrm{rot}_{P;a}$)
  (c) A reflection across a line L with angle b ($\mathrm{ref}_{Q;b}$ for $Q \in L$).
  (d) A glide reflection $T_U \circ\ \mathrm{ref}_{\frac{1}{2}W;b}$

  Note. Even though reflection across L is a glide reflection with $U = 0$, it is useful to distinguish it from the true glides.

- Since these are the **only** affine symmetries, it follows that when we compose two of them, we get another. We've see that:
$$T_V \circ T_W = T_{V+W} \text{ and } \mathrm{rot}_{P;a} \circ \mathrm{rot}_{Q;b} = \mathrm{rot}_{R;a+b}$$
(and that we can solve for R!) Describe the compositions of:
  (i) Reflections across intersecting lines.
  (ii) Reflections across parallel lines.
  (iii) Reflection and rotation around a point on the line.
  (iv) Reflection and rotation around a point not on the line.
  (v) Glide reflections (intersecting and parallel)
  (vi) Glide reflections and rotations.
- Find affine symmetry groups of triangles and parallelograms.
- Two triangles are **congruent** if there is an affine symmetry taking the first to the second. Show that two triangles are congruent if and only if their side lengths are the same.
- Two triangles are **similar** if there is an affine "similarity" (the composition of a dilation with an affine symmetry) taking the first to the second. Show that two triangles are similar if and only if their angle measurements are the same.

Day 10. Wallpaper Groups.

- A wallpaper group is the group of affine symmetries of a *pattern* in the plane that repeats itself in two independent directions.
- Find the wallpaper group for graph paper (pattern of squares).
  Note. The Wiki page for wallpaper groups explains how to indicate rotations, reflections and glide reflections on a single square to indicate all the affine symmetries.
- Find the wallpaper group for equilateral triangle graph paper.
- Find the wallpaper group for rhombus graph paper.
- Find the wallpaper group for rectangular/kite graph paper.
- Find the wallpaper group for regular trapezoidal graph paper.
  Every wallpaper group is a **subgroup** of the group of affine symmetries of either square or triangular graph paper.
  One can "break" the symmetry to find the subgroup by drawing a figure inside the square (or triangle).
- Break symmetry to find the 17 different wallpaper groups(!)
- Explain the names of the groups.
- Investigate the frieze groups.

Further Possibilities.

- Art and wallpaper.
- Complex Möbius transformations.
- Real projective transformations.
- Projective geometry (axioms and properties).
- Hyperbolic geometry (with symmetries).See Escher.
- The Platonic solids and Schläfli four-dimensional solids.
  (What are their symmetry groups?)
- Coxeter reflection groups and their Dynkin diagrams.
- Wallpaper and frieze groups in three dimensions.
  (What are the symmetries of cubical graph paper?)
- Penrose tilings.