Math 4800/6080. Week One Starter Problem

Consider the polynomial equation:
$$x^2 - 2y^2 = 1$$

What can you say about:

(1) The real solutions to this equation (as a subset of the plane $\mathbb{R}^2$)?

(2) The rational solutions
(i.e. real solutions both of whose coordinates are rational numbers)?

(3) The integer solutions?

(4) The complex solutions?

(5) Solutions consisting of pairs of elements of a finite field $\mathbb{F}_p$
(p is a prime)?

Follow-up Problems for Homework.

Answer the same 5 questions for the polynomial equations:

$$x^2 - 4y^2 = 1$$

$$x^2 + y^2 = 25$$

$$x^2 + y^2 = 3$$

Spoiler Alert! The following identity turns out to be key:
$$\left(\frac{1 - 2t^2}{1 + 2t^2}\right)^2 - 2\left(\frac{2t}{1 + 2t^2}\right)^2 = 1$$

(Check it!)

The identity comes from geometry, as I will explain.