Polynomials.
Math 4800/6080 Project Course

2. The Plane.

"Boss, boss, ze plane, ze plane!"

–Tattoo, Fantasy Island

The points of the plane $\mathbb{R}^2$ are ordered pairs (x, y) of real numbers. We'll also use vector notation to denote a point of the plane:

$$\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$$

In high school geometry, we talk about congruences and similarities. These are examples of *affine linear transformations* of the plane.

Definition. A **general linear transformation** is an invertible map:

$$A : \mathbb{R}^2 \to \mathbb{R}^2$$

with the following two properties:

$$A\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = A\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + A\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$$

for all pairs of vectors, and

$$A\left(\alpha \begin{bmatrix} x \\ y \end{bmatrix}\right) = \alpha A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

for all scalars $\alpha \in \mathbb{R}$. In particular, A fixes the origin $(0, 0)$.

A linear transformation is realized by *matrix multiplication:*

$$A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where the matrix is computed via:

$$A\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a \\ c \end{bmatrix} \text{ and } A\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} b \\ d \end{bmatrix}$$

Examples. The *rotation* counter-clockwise (around the origin) by the angle θ is given by:

$$R\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

A rotation is a *rigid motion* of the plane, meaning that it does not change the distances between two points (or the angles between two vectors). It also preserves "orientation," meaning that a clock would measure clockwise in the same way before and after a rotation.

A *reflection* is another rigid motion. The basic reflections are:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

across the y-axis and x-axis, respectively. Reflecting across the x-axis, followed by rotating by θ gives rise to another reflection:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$$

this time across the line given (in polar coordinates!) by:

$$r = \frac{\theta}{2}$$

A little exercise with trigonometry angle addition formulas gives:

$$\begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{bmatrix} \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

from which we verify the (intuitively obvious) fact that rotating by θ_1 and then by θ_2 results in a rotation by $\theta_1 + \theta_2$. In particular, rotations commute with each other. On the other hand, reflecting across $r = \theta_1/2$ followed by reflecting across $r = \theta_2/2$ gives rise to:

$$\begin{bmatrix} \cos(\theta_2) & \sin(\theta_2) \\ \sin(\theta_2) & -\cos(\theta_2) \end{bmatrix} \begin{bmatrix} \cos(\theta_1) & \sin(\theta_1) \\ \sin(\theta_1) & -\cos(\theta_1) \end{bmatrix} = \begin{bmatrix} \cos(\theta_2-\theta_1) & -\sin(\theta_2-\theta_1) \\ \sin(\theta_2-\theta_1) & \cos(\theta_2-\theta_1) \end{bmatrix}$$

which is a **rotation** by the difference of the two angles $\theta_2 - \theta_1$. In particular, the two reflection matrices do **not** (usually) commute since rotating by $\theta_1 - \theta_2$ is the inverse of rotating by $\theta_2 - \theta_1$.

Definition. A set of linear transformations is a *group* (or class) if it is closed under compositions and taking inverses.

In our examples,

(a) The set of rotations (by an angle $0 \le \theta < 2\pi$) is a group, but

(b) The set of reflections across the lines $r = \theta/2$ is not a group, since the composition of two reflections is a rotation, not a reflection. However,

(c) The **union** of the sets of rotations and reflections is a group.

Remark. To complete a verification of (c), you would need to check that a rotation followed by a reflection is a reflection, and also that a reflection followed by a rotation is a reflection. I encourage you to check these and also to verify the two matrix multiplications above.

Definition. A *translation* of $\mathbb{R}^2$ is a mapping:

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

given by:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

Important Remark. A translation is **not** a linear transformation and cannot be described by matrix multiplication. However, a translation is a rigid motion, since it clearly does not change the lengths of vectors or the angles between them. It also, like rotations (and unlike reflections) preserves orientation.

Congruences are rigid motions of the plane that preserve orientation. Each of these can be expressed as a rotation followed by a translation.

$$C\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

But a congruence may be more intuitive if the translation is done first, followed by the rotation. Linear algebra comes to our rescue:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}\right) =$$

$$= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

is **also** a rotation followed by a translation (but not by the same vector!) One can check that a composition of congruences is a congruence and that the inverse of a congruence is a congruence. They are a group.

Debate. Some might argue that a planar shape is congruent to its "mirror image" under a reflection. For this to be so, we need to enlarge our group of congruences to include reflections followed by translations.

Similarities are like congruences, except instead of fixing lengths of line segments, all segments are magnified by a fixed scaling factor $\alpha \neq 0$. This is accomplished by multiplying the rotation matrix by a scaling matrix:

$$S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

A similarity that fixes the origin might be called a *rotation with scaling*:

$$S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Since a composition of similarities will scale lengths by the **product** of the α's, it follows that similarities with a fixed scaling factor are not a group (unless they are congruences). The similarities of all possible (positive) scaling factors do, however, form a group.

General Linear Transformations scale **areas**, rather than lengths:

$$A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

has an area scaling factor equal to the absolute value of the *determinant*

$$\left|\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right| = |ad - bc|$$

(the sign of the determinant determines whether or not orientation is preserved by the transformation). Thus, it is in particular required that the determinant not be zero. These form a group. It is important to keep in mind that a linear transformation may **not** preserve angles, so circles may turn into ellipses, angles of triangles may change, etc.

A general linear transformation followed by a translation is an *Affine (General) Linear Transformation*:

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

These are, for now, our most general transformations of the plane.

As a model for all the other claims we've made, let's prove:

Proposition. The set of Affine Linear Transformations is a group.

Proof. Given two affine linear transformations:

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

and

$$G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

The composition:

$$G \circ F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \cdot \left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$= \left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \cdot \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} + \left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$$

is again an affine linear transformation, and the inverse of F is:

$$F^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}^{-1}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}^{-1}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

Examples of interesting general linear transformations:

$$\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + sy \\ y \end{bmatrix}$$

is a *shear*. It fixes points on the x-axis, and fixes the y-coordinates of **any** point, but it shoves the points of a given y-coordinate to the right (or left) by the fixed multiple s of the y-coordinate.

A *distortion* has different scaling factors in the x and y directions:

$$\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ux \\ vy \end{bmatrix}$$

It is not hard to see that **every** linear transformation with positive determinant (preserving orientation) can be decomposed into a rotation followed by a shear followed by a distortion.

The Projective Plane is a way of "completing" the plane with points at infinity. Projective transformations of the projective plane will be an even larger group of transformations that will be our most general transformations. We will introduce both the projective line and the projective plane simultaneously (to contrast them). The astute reader will notice that there is a pattern here that can be extended to give projective spaces of all dimensions.

Definition. (i) The **projective line** $\mathbb{RP}^1$ is the set of all the lines through the origin in the plane $\mathbb{R}^2$.

(ii) The **projective plane** $\mathbb{RP}^2$ is the set of all the lines through the origin in space $\mathbb{R}^3$.

First Remark. Every line through the origin intersects the *unit sphere* in exactly two points. The unit sphere in the plane is more commonly referred to as the *unit circle*:

$$S^1 := \{(x, y) \; | x^2 + y^2 = 1\}$$

whereas in three space it is what we usually think of as a sphere:

$$S^2 := \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$$

The fact that every line meets the sphere in two (antipodal) points means that there are surjective maps:

$$a_1 : S^1 \to \mathbb{RP}^1 \quad \text{and} \quad a_2 : S^2 \to \mathbb{RP}^2$$

with the property that the inverse image of any point consists of the two "poles" of the intersection of the line with the sphere. These maps make $\mathbb{RP}^1$ and $\mathbb{RP}^2$ into "topological manifolds."

The topological manifold $\mathbb{RP}^1$ is easy to describe. It is also a circle. In fact, the circle is the only connected, one-dimensional closed manifold. In this case, walking along $\mathbb{RP}^1$ is just the same as walking along the unit circle, except that you return to the start after π radians, rather than after 2π radians. Interestingly, the manifold $\mathbb{RP}^2$ is **not** a sphere, as we will see.

Second Remark: Ratios give coordinates of points of $\mathbb{RP}^1$ and $\mathbb{RP}^2$.

For $\mathbb{RP}^1$, these are ordinary ratios:

$$(x : y) \neq (0 : 0) \text{ with } (x : y) = (\alpha x : \alpha y) \text{ for all } \alpha \neq 0$$

Recall that ratios are almost like fractions, except that ratios $(x : 0)$ are allowed, while fractions are not allowed a 0 in the denominator. Only the ratio $(0 : 0)$ is not permitted.

In the case of $\mathbb{RP}^2$, the ratios are "triple" ratios:

$$(x : y : z) \neq (0 : 0 : 0) \text{ with } (x : y : z) = (\alpha x : \alpha y : \alpha z)$$

Special Ratios. The ratios

$$(x : 1) \text{ and } (1 : y)$$

determine two *embeddings* of the real line in $\mathbb{RP}^1$;

$$f : \mathbb{R} \to \mathbb{RP}^1;\ f(x) = (x : 1) \text{ and } g : \mathbb{R} \to \mathbb{RP}^1;\ g(y) = (1 : y)$$

since two ratios $(x : 1)$ and $(x' : 1)$ are different if $x \neq x'$. These two embeddings cover $\mathbb{RP}^1$, with each one missing a different point.

Similarly, the ratios:

$$(x : y : 1), \quad (x : 1 : z) \text{ and } (1 : y : z)$$

determine *three* embeddings of $\mathbb{R}^2$ that cover the projective plane:

$$f(x, y) = (x : y : 1),\ g(x, z) = (x : 1 : z) \text{ and } h(y, z) = (1 : y : z)$$

though each of these embeddings misses a copy of $\mathbb{RP}^1$. For example:

$$(*)\ \mathbb{RP}^2 = \{(x : y : 1)\} \cup \{(x : y : 0)\}$$

and the latter set is identified in the obvious way with $\mathbb{RP}^1$.

All these embeddings may be realized geometrically.

In the case of $\mathbb{RP}^1$, a line through the origin in $\mathbb{R}^2$ intersects the line $y = 1$ at a point $(x, 1)$ (unless it is the x-axis). Similarly, it intersects $x = 1$ at a point $(1, y)$ unless it is the y-axis. In the case of $\mathbb{RP}^2$, the lines through the origin intersect the planes $z = 1, y = 1$ and $x = 1$, respectively in points $(x, y, 1)$, $(x, 1, z)$ and $(1, y, z)$ unless the lines are contained in the xy-plane, xz-plane and yz-plane, respectively.

Focusing on the union $(*)$ above, we may consider the **closure**

$$\overline{S} \subset \mathbb{RP}^2$$

of a subset $S \subset \mathbb{R}^2$ by adding points from the $\mathbb{RP}^1$ "at infinity."

Example: (a) The closure of the line $y = mx + b$ is the single point:

$$(x : mx : 0) = (1 : m : 0)$$

at infinity. This is because the projective coordinates of the "finite" points of the line (with the exception of the y-intercept) are:

$$(x : mx + b : 1) = (1 : m + b/x : 1/x)$$

and the limit as $x \to \pm\infty$ is the point $(1 : m : 0)$. This means that following the line in one direction to infinity will "flip" to the opposite end of the line. The closure of the line is a copy of $\mathbb{RP}^1$!

(b) Consider the region bounded above by:

$$\max\{y = -x, y = 1, y = x\}$$

and bounded below by

$$\min\{y = x, y = -1, y = -x\}$$

The points to add to this bow-tie shaped region to obtain its closure in $\mathbb{RP}^2$ is an interval of points at infinity:

$$\{(1 : m : 0) \mid -1 \le m \le 1\}$$

If you were to walk along the positive x-axis (growing linearly taller and moving exponentially faster in proportion to your distance from the origin), you would reach infinity on the right and then flip **upside down** as you return from infinity on the left. This means that the closure of the bow-tie is a copy of the **Möbius strip**, which is a "one-sided" surface that cannot be embedded in a sphere. Moreover, the complementary region to the bow-tie closure is a topological disk, realizing the projective plane as a "capped" Möbius strip (the edge of the Möbius strip is a circle!) which cannot be placed into $\mathbb{R}^3$. Thus $\mathbb{RP}^2$, unlike $\mathbb{RP}^1$, is not another sphere, but rather a (non-orientable) surface of a different topological type.

Finally, we consider the *projective linear transformations*. As with general linear transformations, these are matrices, but of size one larger than the dimension of the projective space:

Projective Transformations of $\mathbb{RP}^1$ are 2×2 Invertible Matrices:

$$A : \mathbb{RP}^1 \to \mathbb{RP}^1;\quad A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

This is well-defined because matrix multiplication takes lines through the origin to lines through the origin. It is **different** from a general linear transformation of the plane because the vectors of the matrix multiplication are ratios:

$$\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha x \\ \alpha y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So scaled transformations are the **same** projective transformations.

Recall that an affine linear transformation of the real line is:

$$F(x) = ax + b$$

where b is the translation, and a is the scaling factor. Consider the projective transformation of the projective line:

$$\overline{F}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For points of $\mathbb{R}$ (with coordinates $(x : 1)$) this is the **same** as F:

$$\overline{F}\left(\begin{bmatrix} x \\ 1 \end{bmatrix}\right) = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} ax + b \\ 1 \end{bmatrix}$$

and the additional point at infinity is fixed:

$$\overline{F}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

In other words, the affine transformations of the line are a subgroup of the projective transformations of the projective line. This is also the case with the projective plane, except that something interesting happens there at infinity:

Projective Transformations of $\mathbb{RP}^2$ are 3×3 Invertible Matrices:

$$A : \mathbb{RP}^2 \to \mathbb{RP}^2; \quad A\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

As with the projective line, a matrix and a scalar multiple of it define the **same** projective linear transformation, and also as with the projective line, an affine linear transformation:

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

can be extended to the projective linear transformation of $\mathbb{RP}^2$:

$$\overline{F}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} a_1 & b_1 & x_1 \\ c_1 & d_1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This transformation is just F on the points of $\mathbb{R}^2$:

$$\overline{F}\left(\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}\right) = \begin{bmatrix} a_1 & b_1 & x_1 \\ c_1 & d_1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} a_1x + b_1y + x_1 \\ c_1x + d_1y + y_1 \\ 1 \end{bmatrix}$$

but interestingly, unlike the previous case, this extended transformation moves around the points at infinity via a projective transformation of the projective line of points at infinity:

$$\overline{F}\left(\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a_1 & b_1 & x_1 \\ c_1 & d_1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} a_1x + b_1y \\ c_1x + d_1y \\ 0 \end{bmatrix}$$

Permutations are another important group of transformations.

In the projective line case, these are:

$$r = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ with } r^2 = e$$

The transformation r switches the x and y coordinates, hence:

$$r\left(\begin{bmatrix} x \\ 1 \end{bmatrix}\right) = \left(\begin{bmatrix} 1 \\ x \end{bmatrix}\right)$$

so in particular, r switches the two embeddings of $\mathbb{R}$ in $\mathbb{RP}^1$.

In the projective plane case, there are six permutations:

$$e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad c^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$r_{xy} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad r_{xz} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad r_{yz} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

which permute the three embeddings of $\mathbb{R}^2$ in the projective plane.

An Application. We will use the permutation transformations to "swing points in from infinity." In other words, given a subset $\overline{S} \subset \mathbb{RP}^2$, we may apply permutation transformations to $\overline{S}$ to move points at infinity into the plane $\mathbb{R}^2$, and then examine them.