Categories, Symmetry and Manifolds

Math 4800, Fall 2020

6. Representations. We discuss the category of **representations** of a group G.

Definition 6.1. A **representation** is an action of a group G on a vector space. In other words a representation is a homomorphism

$$\rho : G \to \text{ Aut}(V) \text{ to the group of symmetries of } V \text{ in the category } \mathfrak{Vec}_F$$

That is, each $\rho(g) : V \to V$ is a symmetry of the vector space V satisfying:

$$\rho(g_1 g_2)(v) = \rho(g_1) \circ \rho(g_2)(v) \text{ and } \rho(1_G) = 1_V \text{ is the identity symmetry}$$

When $V = F^n$, these symmetries are expressed as $n \times n$ matrices.

Example 1. A representation of $G = \{\pm 1\}$ consists of $\rho(1) = \text{id}_V$ and $\rho(-1) = \sigma$ where $\sigma \in \text{Aut}(V)$ is any symmetry such that $\sigma \circ \sigma = \text{id}_V$. For example

(a) $\rho(1) = I_2$ and $\rho(-1) = I_2$ is a "trivial" representation of G on $\mathbb{R}^2$.

(b) $\rho(1) = I_2$ and $\rho(-1) = -I_2$ is also a representation of G on $\mathbb{R}^2$, as well as

(c) $\rho(1) = I_2$ and any reflection across a line through the origin:

$$\rho(-1) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$$

Remarks. (i) Instead of writing $\rho(g)(v)$, one writes gv when ρ is understood.

(ii) As in the example above, when a group G is given in terms of generators and relations, then a representation ρ is specified by choosing symmetries $\rho(g)$ for each generator so that the group relations hold among the chosen symmetries.

Example 2. Transpositions $g_1 = (1\ 2)$ and $g_2 = (2\ 3)$ generate S_3 with relations:

$$g_1^2 = \text{id}, g_2^2 = \text{id and } (g_1 g_2)^3 = \text{id}$$

Letting

$$\rho(g_1) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \rho(g_2) = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

we check that:

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}^2 = I_2 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}^2$$

and

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \text{ satisfies } \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}^3 = I_2$$

so this determines a two-dimensional representation of S_3. For example,

$$\rho((1\ 2\ 3)) = \rho((1\ 2)(2\ 3)) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \text{ and}$$

$$\rho((1\ 3)) = \rho((1\ 2)(2\ 3)(1\ 2)) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Example 3. A pair of one-dimensional representations of S_n:

(a) The trivial representation $\rho_{tr}(\sigma) = 1$ for all $\sigma \in S_n$ and

(b) The sign representation $\rho_{sgn}(\sigma) = \text{sgn}(\sigma)$ for all $\sigma \in S_n$.

Definition 6.2. Given representations of G on vector spaces V and W, then

$$f : V \to W \in \hom(V, W)$$

is **G-linear** if $f(gv) = gf(v)$ for all $g \in G$ and $v \in V$.

This determines the category $\mathfrak{G}Rep_F$ of G-representations with fixed scalar field F:

- The objects of $\mathfrak{G}Rep_F$ are G-representations on vector spaces over F.
- The morphisms of $\mathfrak{G}Rep_F$ are G-linear maps of vector spaces over F..

Definition 6.3. Given a representation ρ of G on V,

(i) A subspace $U \subset V$ is **invariant** if $gu \in u$ for all $u \in U$.

(ii) ρ is **irreducible** if the only invariant subspaces of V are $\{0\}$ and V.

The Basic Problem. To classify the irreducible representations of a group G.

Examples. (a) In Examples (1a) and (1b), every subspace $U \subset \mathbb{R}^2$ is invariant.

(b) In Example (1c), there are two invariant subspaces: the line $y = \tan(\theta)x$ of vectors with eigenvalue 1 and the perpendicular line $y = -\cot(\theta)x$ of vectors with eigenvalue -1 for the given reflection matrix.

(c) The permutation representation of S_n on F^n given by:

$$\rho_{per}(\sigma)(e_i) = e_{\sigma(i)}$$

has invariant one-dimensional and $n-1$-dimensional subspaces:

$$U_1 = \langle e_1 + \cdots + e_n \rangle \text{ and}$$
$$U_{n-1} = \langle e_1 - e_2, ..., e_{n-1} - e_n \rangle$$

The former is clearly invariant, and latter is invariant because of "telescoping":

$$\rho_{per}(\sigma)(e_i - e_{i+1}) = e_{\sigma(i)} - e_{\sigma(i+1)}$$

and each $e_j - e_k = (e_j - e_{j+1}) + (e_{j+1} - e_{j+2}) + \cdots + (e_{k-1} - e_k) \in U_{n+1}$.

Remark. An invariant subspace $U \subset V$ is itself a representation of G. Thus, for example, the two invariant lines for the reflection in Example (1c) are the representations: ρ_{tr} and ρ_{sgn}, respective, of the group $C_2 = S_2$ (from Example 3). One can check that Example 2 is the representation $U_2 \subset F^3$ for the group S_3.

Definition 6.4. A **character** of G is a one-dimensional complex representation:

$$\chi : G \to \mathbb{C}^* = \mathrm{Aut}(\mathbb{C}^1)$$

Examples include the representations ρ_{tr} and ρ_{sgn} of S_n and the n characters:

$$\chi_m : C_n \to \mathbb{C}^*;\ \chi_m(x) = \zeta^m \text{ for } \zeta = e^{\frac{2\pi i}{n}}$$

(including the trivial character χ_0) of the cyclic group $C_n = \{1, x, x^2, ..., x^{n-1}\}$.

Note. Characters, being one-dimensional, are irreducible complex representations.

The real two-dimensional rotations of C_n corresponding to χ_m are:

$$\rho_m(x) = \begin{bmatrix} \cos(2\pi m/n) & -\sin(2\pi m/n) \\ \sin(2\pi m/n) & \cos(2\pi m/n) \end{bmatrix}$$

and these are irreducible real representations, since they have no invariant lines.

In contrast, we have the following feature of **complex** representations:

Proposition 6.5. If G is an **abelian** group, then every irreducible complex representation of G is one-dimensional, i.e. a character.

Proof. Let $\rho : G \to \mathrm{Aut}(\mathbb{C}^n)$. We show that the commuting matrices:

$$\rho(g) = A_g \text{ for } g \in G$$

all share a common eigenvector. The line spanned by one such eigenvector is then an invariant subspace for the representation ρ (and a character of the group A).

Select $g \in G$ and let $v \in \mathbb{C}^n$ be an eigenvector of A_g with eigenvalue $\lambda \in \mathbb{C}$. Select another $h \in G$. Then because A_g and A_h commute, we have:

$$A_g(A_h v) = A_h(A_g v) = A_h(\lambda \cdot v) = \lambda A_h(v)$$

so $A_h v$ is **another** eigenvector for A_g with eigenvalue λ. View:

$$A_h : V_\lambda \to V_\lambda \text{ as a symmetry of the } \lambda\text{-eigenspace of } A_g$$

Then A_h has an eigenvector in V_λ with eigenvalue μ which is a shared eigenvector. Continue this process to conclude that any finite number of commuting matrices share an eigenvector. But this also applies to an infinite number of commuting matrices, reasoning by induction on the dimension of the shared eigenspaces. □

Example. (a) Consider the "cycle" representation of C_n on $\mathbb{C}^n$ given by:

$$\rho_{cyc}(x)(e_i) = e_{i+1} \text{ for } i < n \text{ and } \rho_{cyc}(e_n) = e_1$$

Then:

$$A_x = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

has a basis $v_1, ..., v_n$ of eigenvectors (hence invariant lines for C_n) given by:

$$v_m = e_1 + \zeta^m e_2 + \zeta^{2m} e_3 + \cdots + \zeta^{(n-1)m} e_n \text{ (with eigenvalue } \zeta^m)$$

and, in particular, $v_n = e_1 + \cdots + e_n$. Notice that each of the invariant lines:

$$(\langle v_m \rangle, \rho_{cyc}) \text{ is a copy of the character } \chi_m \text{ defined above!}$$

(b) Consider the representation of $(\mathbb{C}, +, 0)$ on $\mathbb{C}^2$ given by:

$$\rho(z) = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$$

This has **one** invariant subspace, the line spanned by the joint eigenvector e_1. Notice that when $z \neq 0$ in this example, the matrices are not semi-simple.

Definition 6.6. (i) A vector space V is a **direct sum**:

$$V = U_1 \oplus \cdots \oplus U_n$$

of subspaces $U_i \subset V$ if every vector $v \in V$ has a unique expression:

$$v = u_1 + \cdots + u_n \text{ for vectors } u_i \in U_i$$

(ii) A representation (V, ρ) is a **direct sum**

$$V = U_1 \oplus \cdots \oplus U_n$$

of sub-representations if the U_i are *invariant* subspaces of V.

Any subspace U of a vector space V has many "complementary" subspaces W with $V = U \oplus W$. Indeed, any basis $\{u_1, ..., u_m\}$ for U can be extended to a basis $\{u_1, ..., u_m, v_{m+1}, ..., v_n\}$ for V, and then we may choose $W = \langle v_{m+1}, ..., v_n\rangle$. When $V = \mathbb{R}^n$ or $\mathbb{C}^n$, however, the dot (or standard Hermitian) inner product gives a canonical *orthogonal* complement to U:

$$\mathbb{R}^n : U^\perp = \{v \in V \mid u \cdot v = 0 \text{ for all } u \in U\}$$

$$\mathbb{C}^n : U^\perp = \{v \in V \mid \langle u, v\rangle = 0 \text{ for all } u \in U\}$$

Complements are much rarer for *invariant* subspaces U of a G-representation. For example, in the representation of $(\mathbb{C}, +, 0)$:

$$\rho(z) = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$$

there is only one invariant line $U = \langle e_1\rangle$ which has **no** invariant complement.

Proposition 6.7. If G is finite and $\rho : G \to \mathrm{Aut}(\mathbb{C}^n)$ is a complex representation, every invariant subspace $U \subset \mathbb{C}^n$ has an invariant complement.

Proof. The idea is to construct a Hermitian inner product on $\mathbb{C}^n$ that is G-invariant by averaging over the group, and then to take the orthogonal complement with respect to this averaged inner product. Let:

$$\langle u, v\rangle_G = \frac{1}{|G|}\sum_{g\in G}\langle gu, gv\rangle$$

This is rigged so that:

$$\langle u, v\rangle_G = \langle hu, hv\rangle_G \text{ for all } h \in G$$

and if $v \neq 0$, then

$$\langle v, v\rangle_G = \frac{1}{|G|}\sum_{g\in G}|gv|^2 > 0$$

i.e. $\langle *, *\rangle_G$ is positive definite, and also Hermitian, i.e.

$$\langle u, v\rangle_G = \overline{\langle v, u\rangle}_G \text{ and } \langle c_1u_1 + c_2u_2, v\rangle_G = c_1\langle u_1, v\rangle_G + c_2\langle u_2, v\rangle_G$$

It follows that if U is an invariant subspace of $\mathbb{C}^n$, then $U^\perp$ is complementary, when $U^\perp$ is defined in terms of this G-invariant Hermitian inner product, and

$$\langle u, w\rangle_G = 0 \text{ for all } u \in U \Rightarrow \langle u, hw\rangle_G = \langle h^{-1}u, w\rangle_G = 0 \text{ for all } u \in U$$

so $U^\perp$ is also invariant. □

Example. Consider the action of $G = \{\pm 1\}$ on $\mathbb{C}^2$ with:

$$\rho(1) = I_2 \text{ and } \rho(-1) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

Then $U_1 = \langle e_1\rangle$ is an invariant subspace, and from:

$$\langle e_1, e_1\rangle_G = \frac{1}{2}(\langle e_1, e_1\rangle + \langle -e_1, -e_1\rangle) = 1 \text{ and}$$

$$\langle e_1, e_2\rangle_G = \frac{1}{2}(\langle e_1, e_2\rangle + \langle -e_1, e_1 + e_2\rangle) = -\frac{1}{2}$$

we conclude that:

$$\langle e_1, e_1 + 2e_2\rangle_G = 0$$

i.e. $U_1^\perp = \langle e_1 + 2e_2\rangle$ (which is an eigenvector of $\rho(-1)$ with eigenvalue one!)

Corollary 6.8. Every complex representation (V, ρ) of a finite group G decomposes as a direct sum of irreducible representations:

$$V = U_1 \oplus \cdots \oplus U_m$$

Proof. If V is irreducible, then the Corollary is trivially true. Otherwise V has an invariant subspace $U \subset V$, and then V decomposes as $V = U \oplus U^\perp$ via the Proposition. Then we apply the Proposition to U and $U^\perp$ individually and proceed until we get the desired decomposition. □

Corollary 6.9. If $A \in \mathrm{Aut}(\mathbb{C}^n)$ and $A^d = I_n$, then A is semi-simple, i.e. $\mathbb{C}^n$ has a basis of eigenvectors of the matrix A.

Proof. Let $G = C_n$, and $(\mathbb{C}^n, \rho)$ be the representation determined by $\rho(x) = A$. Then A has an eigenvector v_1 with eigenvalue λ_1. Moreover, since:

$$A^m v_1 = \lambda_1^m v_1$$

it follows that v_1 is an eigenvector for all powers of A, and $\langle v_1 \rangle$ is an invariant subspace of $\mathbb{C}^n$. Let $U^\perp \cong \mathbb{C}^{n-1}$ be the complementary invariant subspace from the Proposition. Then by induction on n, $A : U^\perp \to U^\perp$ has a basis $v_2, ..., v_n$ of eigenvectors, and so $v_1, v_2, ..., v_n$ is a basis of eigenvectors of $\mathbb{C}^n$. □

Corollary 6.10. If $(\mathbb{C}^n, \rho)$ is a representation of a finite group G, then each

$$\rho(g) = A_g \text{ is semi-simple}$$

Proof. Each of these matrices has order d for some d. □

An Irreducible Complex Representation. Let D_{2n} be the dihedral group, generated by two elements g_1 and g_2 with relations:

$$g_1^2 = 1, g_2^2 = 1 \text{ and } (g_1 g_2)^n = 1$$

We've seen in §4 that one representation of D_{2n} is given by the symmetries of the regular n-gon (with vertices at $(\cos(2\pi m/n), \sin(2\pi m/n))$ and reflections:

$$\rho(g_1) = \begin{bmatrix} \cos(2\pi/n) & \sin(2\pi/n) \\ \sin(2\pi/n) & -\cos(2\pi/n) \end{bmatrix} \text{ and } \rho(g_2) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

with

$$\rho(g_1 g_2) = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}$$

Using the same matrices, we promote this to a representation of D_{2n} on $\mathbb{C}^2$.

This complex representation of D_{2n} is irreducible since any invariant line would be spanned by a common eigenvector for $\rho(g_1)$ and $\rho(g_2)$, and by virtue of being reflections across different lines of symmetry, these share **no** common eigenvectors. Specifically, the invariant lines for these transformations are $(y = \tan(2\pi/n)x)$ and $(y = -\cot(2\pi/n))$ for the matrix $\rho(g_1)$ and $(x = 0)$ and $(y = 0)$ for $\rho(g_2)$.

Contrast this with the two-dimensional complex representation of the cyclic group C_n given by the rotation matrix:

$$\rho(x) = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}$$

which has two invariant (complex) lines $\langle e_1 + ie_2 \rangle$ and $\langle e_1 - ie_2 \rangle$ that correspond to the two characters χ_{-1} and χ_1, respectively.

Finally, consider the "pair of" two-dimensional representations of the group S_3:

$$\rho(g_1) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \rho(g_2) = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

(from Example 2) and:

$$\tau(g_1) = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \text{ and } \tau(g_2) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

viewing S_3 as the dihedral group D_6 acting on the equilateral triangle.

I claim that these are the **same** representation of S_3, with the different matrix representations an artifact of the choice of different bases for $\mathbb{C}^2$. In other words, we seek a single "change of basis" matrix B such that:

$$B^{-1} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} B = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

and

$$B^{-1} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

It is easiest to work with the second equation, and to recall that because the change of basis B converts to a *diagonal* matrix, then:

$$B = [v_1 \ v_2] \text{ where } \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} v_1 = v_1 \text{ and } \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} v_2 = -v_2$$

i.e. v_1 and v_2 are eigenvectors with $+1$ and -1 eigenvalues. A bit of fiddling gives:

$$v_1 = \lambda_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } v_2 = \lambda_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ for } \lambda_1, \lambda_2 \in \mathbb{C}^*$$

and then plugging in for B we find that setting $\lambda_2/\lambda_1 = \sqrt{3}$ gives

$$B^{-1} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} B = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$