


4800-20

Character Tables

Last time: G finite gp.

$$\mathbb{C}[G] = \langle e_g \mid g \in G \rangle \quad \checkmark$$

\uparrow
(basis vectors) $= \left\{ \sum_j c_j e_j \right\}$

$$h \cdot e_g = e_{h \cdot g} \quad \text{defined the action}$$

$$\mathbb{C}[G] = \{ \text{functions } f: G \rightarrow \mathbb{C} \}$$

e_g ^{def} delta functions

$$\delta_g(h) = \begin{cases} 1 & h=g \\ 0 & h \neq g \end{cases}$$

$$f = \sum_{g \in G} f(g) \underset{\cong}{\delta}_g$$



$$h \cdot f = f \circ (left mult by h^{-1})$$

$$h \cdot \delta_g = \delta_{hg}$$

$$[CLG] = \bigoplus_{U \text{ irred.}} (\dim U) \cdot U$$

Problem: Find all the U 's

$$|G| = \sum_U (\dim U)^2$$

Def: The character χ_ρ
of a representation (V, ρ) is:

$$\chi_\rho : G \rightarrow \mathbb{C} \quad \text{defined by} \\ \underline{\chi_\rho(g)} = \text{tr}(\underline{\rho(g)}).$$

Rmk: If $f = \chi: G \rightarrow \mathbb{C}^*$

is a one-dim'l repn then

$$\chi_\chi(g) = \text{tr}(\chi(g)) = \chi(g).$$

i.e. $\chi_\chi = \chi.$

Example: Two-dim' repn. of S_3

$$\rho: S_3 \rightarrow \text{Aut}(\mathbb{C}^2)$$

$$\rho(12) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \rho(23) = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

$$\chi_{\rho}(\text{id}) = \text{tr} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2 \quad (= \dim V)$$

$$\begin{aligned} \chi_{\rho}(12) &= \text{tr} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = 0 & \begin{array}{c} \uparrow \\ \nearrow \\ \square \end{array} \\ \chi_{\rho}(23) &= 0, \quad \chi_{\rho}(13) = 0 & \leftarrow \end{aligned}$$

$$\chi_{\rho}(123) = \text{tr} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = -1$$

$$\begin{aligned} \chi_{\rho}(132) &= \text{tr} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \text{tr} \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \\ &= -1 \end{aligned}$$

$$\text{Rmk: } \chi_f(h) = \chi_f(g h g^{-1})$$

for all $g \in G$.

$$\chi_f(h) = \text{tr}(\rho(h)) \quad B \cdot A \cdot B^{-1}$$

$$\begin{aligned} \chi_f(g h g^{-1}) &= \text{tr}(\rho(g) \cdot \rho(h) \cdot \rho(g)^{-1}) \\ &= \text{tr}(\rho(h)) = \chi_f(h). \end{aligned}$$

Def: A function

$$f: G \rightarrow \mathbb{C} \quad \text{is a } \underline{\text{class}}$$

function if $f(h) = f(g h g^{-1})$ for
all $g, h \in G$.

Note: $\dim \{ \text{class functions} \}$
 $f: G \rightarrow \mathbb{C}$

= # of conjugacy classes

$$\underline{\text{E.g.}} \quad \chi_p \quad \{h\} = \{ghg^{-1}\} \\ = G h G^{-1}$$

$$\chi_p([h]) = \begin{cases} 2 & \text{if } [h] = [\text{id}] \\ 0 & \text{if } [h] = [(1\ 2)] \\ -1 & \text{if } [h] = [(1\ 2\ 3)] \end{cases}$$

Thm: Define a Hermitian inner product of the space of class functions: ✓ ✓

$$\alpha, \beta: G \rightarrow \mathbb{C}$$

(constant on conj. classes)

$$[\underline{(\alpha, \beta)} = \frac{1}{|G|} \sum_{g \in G} \alpha(g) \cdot \overline{\beta(g)}]$$

Then the characters of irreps of G form an orthonormal basis for the class functions!!

S

(id) {c(12)} {c(123)} conj. classes
 id c(2) c(123) ↗ represent.

A diagram illustrating three functions, f_1 , f_2 , and f_3 , plotted against the number of elements (labeled '# of elements' on the horizontal axis). The horizontal axis has tick marks at 1, 3, and 2. The vertical axis is labeled 'x'. The functions are defined as follows:

- f_1 is a step function that jumps to 1 at 1 element, stays at 1 until 3 elements, and then drops to 0 at 2 elements.
- f_2 is a sawtooth wave that starts at 1 for 1 element, decreases linearly to -1 for 2 elements, and then increases linearly back to 1 for 3 elements.
- f_3 is a step function that jumps to 1 at 2 elements and stays at 1 until 3 elements.

↑

irreps

$$\chi_{\rho}(g) = \text{tr}(\rho(g))$$

$$(f_1, f_1) = \frac{1}{6} \sum_{g \in G} f_1^2(g) = \frac{1}{6} (1 + 3 \cdot 1 + 21) = 1.$$

$$(f_2, f_2) = \frac{1}{6} (1^2 + 3(-1)^2 + 2 \cdot 1^2) = 1$$

$$(f_1, f_2) = \frac{1}{2} (2^2 + 3 \cdot 0^2 + 2 \cdot (-1)^2) = 1$$

$$\begin{array}{c} \text{C}_3 \\ \equiv \end{array}$$

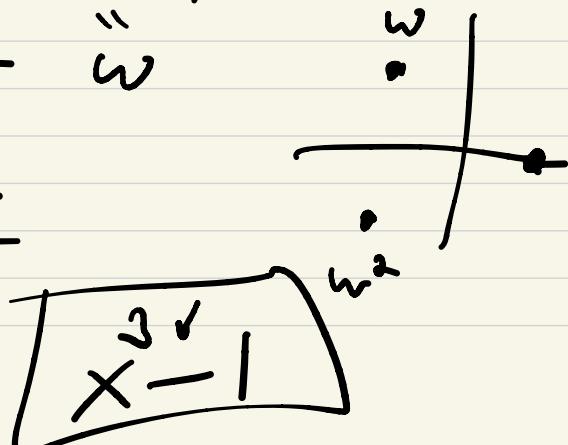
	id	x	x^2	$w^3 = 1$
χ_0	1	1	1	$\leftarrow f_1$
χ_1	1	ω	w^2	$\leftarrow f_2$
χ_2	1	w^2	$w^4 = w$	$\leftarrow f_3$

$$(f_1, f_1) = \frac{1}{3} (1^2 + 1^2 + 1^2) = 1$$

$$(f_2, f_2) = \frac{1}{3} (1^2 + \underbrace{\omega \cdot \overline{\omega}}_{w^2} + \underbrace{w^2 \cdot \overline{w^2}}_{w}) = 1$$

$$(f_1, f_2) = \frac{1}{3} (1^2 + \underbrace{\overline{\omega}}_{w^2} + \underbrace{\overline{w^2}}_{w}) = 0$$

Work the rest out



A_4	id	$(12)(34)$	(123)	(132)
χ_{tr}	1.	3..	4	4
χ_w	1.	1.	w.	w^2
χ_{w^2}	1	1	w^2	w
Tet	3.	-1..	0	0

$$A_4 / K_4 = C_3$$

C

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left\{ \frac{K_4}{\text{id}}, (123)K_4, (132)K_4 \right\}$$

~~Diagram of a square with vertices labeled 1, 2, 3, 4. A diagonal line connects vertex 1 to vertex 3. A counter-clockwise arrow indicates a rotation of $180^\circ = \pi$.~~

$$1 + (-1) + (-1) = -1$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \cos(2\pi/3) & \sin(2\pi/3) \\ \sin(2\pi/3) & -\cos(2\pi/3) \end{pmatrix}$$

$$\begin{pmatrix} \cos(\pi/3) & \sin(\pi/3) \\ \sin(\pi/3) & -\cos(\pi/3) \end{pmatrix}$$

S_4

$$\left[\cancel{S_4 \times F_4 = S_3} \right] \downarrow$$

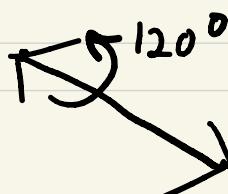
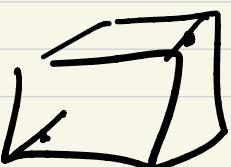
$\begin{matrix} id & c(12) & c(12)c(34) & c(123) & c(1234) \\ 1 & 2 & 3 & 8 & 6 \end{matrix}$

χ_{id}	1.	1	1	1	1
χ_{sgn}	1	-1	1	1	-1

Δ	2	0	2	-1	0
Δ	3	-1	-1.	0	1.

ξ_n	3	1	-1	0	-1
ξ_n	3	1	-1	0	-1

$$(1^2 + 1^2 + 4 + 9 + \underline{\quad} = 24) \quad \left[\begin{smallmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \end{smallmatrix} \right] + \begin{smallmatrix} 2 \\ 1 \\ 1 \end{smallmatrix} \quad \overset{c(12)x}{\boxed{}}$$



$$c(12)$$

Fun Fact:

If $\chi: G \rightarrow \mathbb{C}^*$ is a one-dim ch.

and $\rho: G \rightarrow \text{Aut}(V)$ is

is a rep, then

$$\chi \cdot \rho(g) = \tilde{\chi(g)} \cdot \rho(g)$$

is a representation!

Rank: $\chi_\rho = \chi_{\tilde{\chi} \cdot \rho} \Leftrightarrow \underline{\underline{\rho = \tilde{\chi} \cdot \rho}}$

challenge! A₅

=

and

S₅

$$y = bx + c$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+cy \\ cx+dy \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftrightarrow x$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftrightarrow y$$

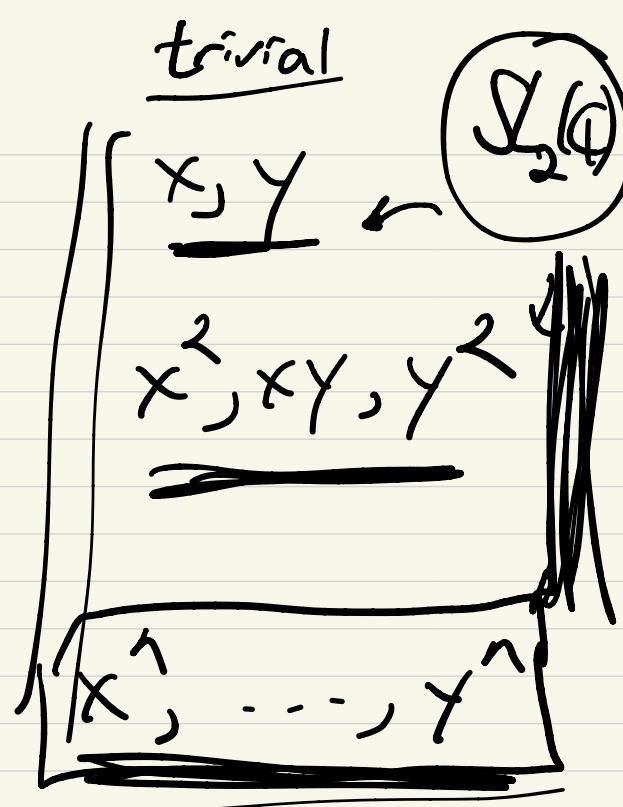
$$\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}^2 \right| = (ax+cy)^2$$

$$\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} xy \\ y^2 \end{pmatrix} \right| = (ax+cy) \times (\text{scratch})$$

$$\begin{matrix} \mathbb{C}^1 \\ \mathbb{C}^2 \end{matrix}$$

$$\begin{matrix} 3 \\ \vdots \\ \vdots \end{matrix} \quad \mathbb{C} = \text{Sym}^2(\mathbb{C}^2)$$

$$\mathbb{C} = \text{Sym}^n(\mathbb{C}^2)$$



Fact: These are all the

irreps of $SL(2, \mathbb{Q})$

$$\boxed{\mathbb{C}[x, y]} = \boxed{\bigoplus_{d=1}^n \mathbb{C}[x, y]_d}$$

$\mathbb{C}[x, y] \quad \bigoplus_{d=1}^n \mathbb{C}[x, y]_d$

$x \rightarrow x^d \quad y \rightarrow y^d$

$SL_2(\mathbb{Q})$ $\langle x, y \rangle$

$(\mathbb{C}L(x,y))$

\cup

(irrep of $SL_2(\mathbb{C})$)

\cup_n
analogue of $(\mathbb{C}LG)$
(for $SL(3,\mathbb{Q})$)

(res
repr. of
 G)

$(\mathbb{C}LG), \rho$) res.

why are $(\mathbb{C}L(x,y))_d$ irreducible

reps of $SL(2,\mathbb{C})$ and ↪

why are there all of the
irred. reps. of $SL(3,\mathbb{Q})$? ↪