Categories of Sets
(Tale of Two Categories)

Subsets of $\mathcal{U}$

$\mathcal{U} = P \mathcal{U}$, any set
Sub category of subsets of $U$

- Objects: subsets of $U$
  $S, T, \ldots \subseteq U$

- Morphisms: set inclusions

$S \subseteq T$

Note:

$\text{hom}(S, T) = \begin{cases} \{\{S, T\}\} & \text{or} \\ \emptyset & \text{if } S \neq T \end{cases}$

$R \subseteq S \subseteq T$ and $S = S$
$U = \emptyset$

**One object:** \( \emptyset \)

**One morphism:** \( \emptyset = \emptyset \)

$U = \{ \emptyset \}$

**Two objects:** \( \emptyset, \emptyset \)

$U = \emptyset$
\[ U = \{ 0, \ldots, 13 \} \]

- 4 objects

\[ \emptyset, \{0\}, \{0, 1\}, \{0, 1, 2\} \]

Initial object: \( \emptyset \)

Final object: \( \{0, 1, 2\} \)
Three operations on objects

\( \cap, \cup, \complement \)

- Intersection
- Union
- Complement

Rmk: \( \text{Sub}_U \)

- The collection of objects of \( \text{Sub}_U \) is a set \( \mathcal{P}(U) \) - power set of \( U \).
Sets

- objects are sets, $S, T, U, \ldots$

- morphisms are functions $\mathbb{f}: S \rightarrow T$

- $\text{hom}(S, T) = \{ f \}$

- $\mathbb{1}_S: S \rightarrow S$
Russell's Paradox:

The collection of sets is not a set.

If $U$ were the set of sets, then

$\forall U \in U$

$x = \{ S \in U \mid S \notin S \}$

$X \subset X \Rightarrow X \notin X \Rightarrow X \notin X$
Picture of

$$4 = \left| \text{hom} (\Sigma 1, 2^3, \Sigma 1, 2^3) \right|$$

Endomorphisms

$$\text{not sym.}$$

$$-1$$

$$f(1) = 1 \quad f(1) = 1 \quad f(1) = 1 \quad f(1) = 1$$

$$f(2) = 2 \quad f(2) = 2 \quad f(2) = 2 \quad f(2) = 2$$

$$= \quad =$$
Def: A symmetry of $X$ (object of a category) is an isomorphism

\[
f : X \rightarrow X
\]

\[f^{-1}
\]

In Sets, the permutations are the symmetries of finite sets.
Definitions:

\[ \mathbb{N} = \{1, \ldots, n\} \]

\[ \mathbb{Z} = \{1\} \]

\[ \mathbb{C} = \{1, 2\} \]

etc.

Every finite set is isomorphic with a unique \( \mathbb{N} \).

\[ \mathbb{N} \rightarrow \mathbb{N}, \ x \mapsto x \]
Counting Problems

1. If \( m < n \), count all the injective functions \( f : [m] \rightarrow [n] \)

\[ f \text{ is injective if } \]
\[ x \neq y \implies f(x) \neq f(y) \]

\[ f \text{ is surjective if } \]
\[ \forall x \in X, \exists y \in [n] \text{ such that } f^{-1}(y) = x \neq \emptyset \]
Answer to (a)

- \( n^m \) functions from \( \mathcal{L}_m \) to \( \mathcal{L}_n \)

- \( n \cdot (n-1) \ldots (n-m+1) \) injective. Easy!

(2) Count the surjective functions from \( \mathcal{L}_n \) to \( \mathcal{L}_n \)

Hard!
The surjective maps

\[ \left\{ \begin{array}{c}
\times \quad 3 \\
\downarrow \\
3 \\
\end{array} \right. \xrightarrow{\cdot} \left. \begin{array}{c}
\times \quad 2 \\
\downarrow \\
\end{array} \right. \]

\[ 2^n - 2 \]

Try: \( m = 3 \)!!
Two Operators on Sets

Given $S, T$, then

$S \times T = \{ (s,t) \mid s \in S \text{ and } t \in T \}$

Cartesian Product

$S \cup T$ disjoint union

and a set \( \emptyset \)
$$\Sigma_{\mathcal{MT}} = \{ (s,i) \mid s \in S \}$$

$$\cup \{ (t,-1) \mid t \in T \}$$

Want: If $\mathcal{MT}$ are finite

$$|S \times T| = |S| \cdot |T|$$

$$|\Sigma_{\mathcal{MT}}| = |S| + |T|$$

($$|\Sigma_{\mathcal{MT}}| = |S| + |T| - |\text{SUS}|$$

($$\text{SUS} \neq S$$).
Study symmetries of finite sets.

Suffice to study symmetries of $\mathbb{Z}_n$.

**Def.** An isomorphism $f: \mathbb{Z}_n \rightarrow S_n$ is called an ordering of $S_n$. 
Transferring symmetries of $\ln$ to symmetries of $s$

\[
\mathcal{O} = \text{symmetry at } \ln
\]

\[
\ln \xrightarrow{f} \text{symmetry at } s
\]

\[
\text{footnote}
\]
\[ \ln(1) = 2 \]

\[ f(1) = f(2) = 1 \]

\[ f(1) = x \]

\[ f(2) = y \]

\[ f \circ f^{-1}(x) = y \]
\[ \sigma_1 \circ \sigma_2 \]

\[ \left( f_{\sigma_1} \circ f_{\sigma_2}^{-1} \right) \left( f_{\sigma_2} \circ f_{\sigma_1}^{-1} \right) \]

= \text{composition of transfers}

\text{transfer of composites}

\[ f_{\sigma_1} \circ f_{\sigma_2}^{-1} \left( f_{\sigma_2} \circ f_{\sigma_1}^{-1} \right)^{-1} \]
Understand symmetries of \([C_n]\).

Permutations of \([C_n]\) have a sign.

Let \(f : [n] \rightarrow [n]\).

**Definition:**

\[
\text{sgn}(f) = \prod_{i<j} \frac{f(j) - f(i)}{j - i}
\]
Example:

\[ f(1) = 2, \quad f(2) = 3, \quad f(3) = 1 \]

\[ \text{sgn}(\mathcal{G}) = \prod_{i<j} \frac{f(i) - f(j)}{j-i} \]

\[ = \frac{f(2) - f(1)}{2-1} \cdot \frac{f(3) - f(1)}{3-1} \cdot \frac{f(3) - f(2)}{3-2} \]

\[ a_f(1) \quad a_f(2) \quad a_f(3) \]
= 1 \cdot -\frac{1}{2} \cdot -2 = +1

Proposition:

(a) If \( f \) is not a symmetry, then \( \text{sgn}(f) = 0 \)

(b) Otherwise \( \text{sgn}(f) = \begin{cases} +1 \\ -1 \end{cases} \)

(c) \( \text{sgn}(f \circ g) = \text{sgn}(f) \cdot \text{sgn}(g) \)