Math 4200-001/Complex Analysis/Fall 2017

Introduction

This is a course on **functions of one complex variable**:

$$f : \mathbb{C} \to \mathbb{C};\ f(z) = u(z) + iv(z)$$

Since the complex numbers live in the plane, we may regard f as:

$$f : \mathbb{R}^2 \to \mathbb{R}^2;\ f(x, y) = (u(x, y), v(x, y))$$

a pair of real-valued functions of two real variables.

Continuity of such functions is the usual notion from real analysis. The function f is continuous if and only if u and v are continuous real functions of two real variables. On the other hand, **differentiability** of the function $f(z)$ requires more than the existence of partial derivatives of u and v as functions of two real variables.

The definition of the derivative at z_0 is the usual one:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

but this limit must not depend on how z approaches the value z_0!

Examples. (i) Constant functions $f(z) = c$ are differentiable, since:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} 0 = 0$$

(ii) The function $f(z) = z$ is also differentiable everywhere, with:

$$f'(z_0) = \lim_{z \to z_0} 1 = 1 \text{ for all } z_0 \in \mathbb{C}$$

(iii) The conjugation function $f(z) = \overline{z}$ in **not** differentiable!

$$\lim_{z \to z_0} \frac{\overline{z} - \overline{z}_0}{z - z_0} = \lim_{z \to z_0} \frac{\overline{z - z_0}}{z - z_0}$$

does depend on how z approaches z_0. For example, if $z - z_0 = h$ is real, then $\overline{z - z_0}/z - z_0 = h/h = 1$, but if $z - z_0 = ih$ is purely imaginary, then $\overline{z - z_0}/z - z_0 = -ih/ih = -1$. They don't agree!

The **Cauchy-Riemann condition** gives additional conditions on the partial derivatives of u and v that are necessary to the existence of a complex derivative. Namely,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

As another example, consider $f(z) = z^2$. This is differentiable, since:

$$f'(z_0) = \lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z \to z_0} z + z_0 = 2z_0$$

(the same reasoning shows that polynomials in z are differentiable).

On the other hand,

$$z^2 = (x + iy)^2 = x^2 - y^2 + i(2xy)$$

so $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Now:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -2y$$

so the Cauchy-Riemann equations check out.

Notice also that taking another derivative gives:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 u}{\partial^2 y} = -\frac{\partial^2 v}{\partial x \partial y}$$

and by the equality of mixed partials, we get the PDE:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ (also satisfied by } v)$$

Such functions are **harmonic** functions of two real variables, which satisfy the "maximum principle:" all the extreme values of a non-constant harmonic function on a domain in $\mathbb{R}^2$ occur on the **boundary**, i.e. all critical points of a harmonic function are saddle points.

In the example above, the functions $x^2 - y^2$ and $2xy$ each have one critical point, which is a saddle point at the origin.

Rational functions and convergent power series are all differentiable. In fact, these are (locally) the *only* differentiable functions:

Analyticity. If $f'(z_0)$ exists, then all derivatives of $f(z)$ at z_0 exist, and $f(z)$ is equal to its Taylor series in some neighborhood of z_0.

Example. The exponential function extends to a complex variable:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

and converges everywhere. Restricting to the imaginary axis:

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \cdots = \cos(\theta) + i\sin(\theta)$$

simply by collecting the terms of the power series! Then:

$$e^z = e^{x+iy} = e^x \cos(y) + i \cdot e^x \sin(y)$$

A crucial tool for proving analyticity (important on its own) is the **Cauchy Integral Formula**, which states that the value of a complex differentiable function at $z_0 \in \mathbb{C}$ is computed by the **line integral**:

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z_0} dw$$

around any closed curve C inside the domain of differentiability of $f(z)$ that "winds" around z_0 once in the counterclockwise direction.

This will allow us, for example, to compute some ordinary calculus integrals like magic with **countour** integration.

Finally, a differentiable function $f(z)$ is **conformal** at a each point $z_0 \in \mathbb{C}$ with $f'(z_0) \neq 0$. This means that the angle between two curves meeting at z_0 is the same as the angle of their images under the mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$. This final remarkable property allows us to visualize these mappings and the image of a small disk around z_0 under a differentiable function with non-zero derivative at z_0.

Our objective in this course is to understand, prove and apply all the elements of the story outlined above. Some of the proofs will be messy, requiring us to approximate curves in the plane with piecewise linear curves, and to consider carefully the **topology** of regions in $\mathbb{R}^2$. On the other hand, the payoff is immense. Complex functions are essential to an understanding of electricity and magnetism, quantum mechanics, probability theory and the even the distribution of prime numbers!