Math 4030-001/Foundations of Algebra/Fall 2017
Final Exam (Two Pages!)

1. (50 points).

Convert the following into quantified math statements and prove them.

(a) The sum of a rational and an irrational number is irrational.

(b) Every finite set of rational numbers has a largest element.

(c) Every real number other than zero has a multiplicative inverse.

(d) The only polynomial with rational coefficients and infinitely many roots is the zero polynomial.

(e) A cubic polynomial with rational coefficients is prime if and only if it has no rational roots.

2. (50 points) Find examples for each of the following.

(a) An equivalence relation on $\mathbb{Z}$ with four equivalence classes.

(b) A cubic polynomial $f(x) \in \mathbb{Q}[x]$ with $f(1) = f'(1) = f''(1) = 0$.

(c) A rational number r with $r^2 > 3$ and $r^2 < 3.000001$.

(d) A polynomial $f(x) \in \mathbb{Q}[x]$ with $f(1+i) = 0 = f(1+2i)$.

(e) A polynomial $f(x) \in \mathbb{Q}[x]$ with $f(\sqrt{2} + \sqrt{3}\, i) = 0$.

3. (20 points) (a) Solve the equation

$$565a + 1001b = 11 \text{ with integers } a, b$$

(b) Solve the equation:

$$(x^4 + 1)f(x) + (x^6 - 1)g(x) = 1 \text{ with polynomials } f(x), g(x)$$

4. (20 points) (a) Find **all** the cube roots of:

$$4\sqrt{2} - 4\sqrt{2}\, i$$

(b) Find **all** the real and complex roots of the polynomial:

$$x^6 + 2x^2 - 3$$

5. (20 points) Find the inverse of $2 + 3\sqrt[3]{2}$ in the number field $\mathbb{Q}[\sqrt[3]{2}]$. In other words, rationalize the fraction:

$$\frac{1}{2 + 3\sqrt[3]{2}}$$

6. (20 points) (a) Prove that the polynomial $x^4 - 2$ is prime in $\mathbb{Q}[x]$. (Make sure you show that it is not a product of a pair of quadratic polynomials with rational coefficients.)

(b) Find the minimal and characteristic polynomials of:

$$1 + \sqrt{2} \in \mathbb{Q}[\sqrt[4]{2}]$$

7. (20 points) Explain how to construct the following numbers or else explain why they cannot be constructed.

(a) $\cos\left(\frac{\pi}{5}\right)$ (b) $\cos\left(\frac{\pi}{9}\right)$ (c) $\sqrt[4]{5}$ (d) $\sqrt{1+\sqrt{2}}$