Infinite Series

Definitions:
1. Sequence: a list of numbers, in order, that follow a pattern
2. \( f: \mathbb{N} \rightarrow \mathbb{R} \)
   - A function that inputs natural numbers and maps them to real numbers
3. Series: the sum of the sequence

Examples of Infinite Series:
1. 1, 2, 3, …  
   Natural Numbers
2. \( \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \)  
   Zeno’s Paradox
3. 1, 1, 2, 3, 5, 8, …  
   Fibonacci Sequence
4. 1, -1, 1, -1, 1, …  
   Alternating Yoyo (class name)
5. \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \)  
   Harmonic Series
6. \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \)  
   Alternating Harmonic Series

Arithmetic Series:
\[ f(n) = f(n - 1) + d \quad \text{Recursive} \]
\[ f(n) = f(0) + dn \quad \text{Explicit} \]
where \( d \) is the common difference

Geometric Series:
\[ f(n) = f(n - 1) \times r \quad \text{Recursive} \]
\[ f(n) = f(0) \times r^n \quad \text{Explicit} \]
where \( r \) is the common ratio

Examples of Arithmetic and Geometric Series:
1. Explicit Arithmetic
   \[ f(n) = 5 + 2n \]
   \{5, 7, 9, …\}
2. Explicit Geometric
   \[ f(n) = 5 \times 2^n \]
   \{5, 10, 20, …\}
Derivation of Geometric Series:

\[ S_n = 1 + r + r^2 + \ldots + r^n \]
\[ rS_n = r + r^2 + r^3 + \ldots + r^{n+1} \]
\[ (1 - r)S_n = (1 + r + r^2 + \ldots + r^n) + (r + r^2 + r^3 + \ldots + r^{n+1}) \]
\[ (1 - r)S_n = 1 - r^{n+1} \]
\[ S_n = \frac{1 - r^{n+1}}{1 - r} \quad \text{where } -1 < r < 1 \text{ as } n \to \infty \]

We know that \( r^{n+1} \to 0 \) as \( n \to \infty \). Therefore,
\[ S_n = \frac{1}{1 - r} \]
\[ S_n = \frac{1 - 0}{1 - r} = \frac{1}{1 - r} \]

Example of Geometric Series:

1. When \( r = \frac{1}{3} \), the series looks like
\[ 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots + \frac{1}{3^n} \]
Therefore,
\[ S_n = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2} \]

2. When \( r = \frac{2}{3} \), the series looks like
\[ 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \ldots \]
Therefore,
\[ S_n = \frac{1}{1 - \frac{2}{3}} = \frac{3}{1} = 3 \]

The Comparison Test:
If you have two positive sequences \( a_n \) and \( b_n \), and sequence \( a_n \) has terms smaller than \( b_n \) and \( \sum b_n \) converges then \( \sum a_n \) converges.
If you have two positive sequences \( a_n \) and \( b_n \), and sequence \( b_n \) has terms larger than \( a_n \) and \( \sum b_n \) diverges then \( \sum a_n \) diverges.

Definition of an Alternating Series:
\[ (-1)^n \times \text{positive series} \]
Example of the Comparison Test:
\[ a_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \]
\[ b_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \]
\[ = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \]
We know that \( \sum b_n \) diverges and \( b_n \) has larger terms than \( a_n \), therefore, \( \sum a_n \) diverges.

Solving Series in Physics:
Physicians will tell you the following,
\[ S = 1 + 2 + 3+ \ldots = -\frac{1}{12} \]
Here is the proof behind it. Given
\[ S_1 = 1 - 1 + 1 - 1+ \ldots \]
\[ S_2 = 1 - 2 + 3 - 4+ \ldots \]
We can add \( S_1 + S_1 \)
\[ S_1 + S_1 = 1 - 1 + 1 - 1+ \ldots \]
\[ + \quad 1 - 1 + 1- \ldots \]
\[ \hline \]
\[ 1 + 0 + 0 + 0+ \ldots \]
We can see that
\[ S_1 + S_1 = 1 + 0 + 0 + 0+ \ldots \]
\[ 2 S_1 = 1 \]
\[ S_1 = \frac{1}{2} \]
We can add \( S_2 + S_2 \)
\[ S_2 + S_2 = 1 - 2 + 3 - 4+ \ldots \]
\[ + \quad 1 - 2 + 3- \ldots \]
\[ \hline \]
\[ 1 - 1 + 1 - 1 + \ldots \]
We can see that
\[ S_2 + S_2 = 1 - 1 + 1 - 1+ \ldots = S_1 \]
\[ 2 S_2 = S_1 \]
\[ S_2 = \frac{1}{4} \]

Now we can subtract \( S - S_2 \)
\[ S - S_2 = 1 + 2 + 3 + 4+ \ldots \]
\[ -1 + 2 - 3 + 4- \ldots \]
\[ \hline \]
\[ 0 + 4 + 0 + 8+ \ldots \]
We can see that
\[ S - S_2 = 4S \]
\[ S - \frac{l}{4} = 4S \]
\[ -\frac{l}{4} = 3S \]
\[ S = -\frac{l}{12} \]