

# Infinite decimal expansions of rational number

Terminating decimal- if you multiply it by a multiple of ten you get a whole number

Repeating decimal - is a decimal number with a digit, or group of digits, that repeat on and on, without end

Every fraction repeats

$$\begin{array}{r}
 \overline{.142857} \\
 7 \sqrt{1.000000} \\
 \underline{.7} \phantom{000000} \\
 30 \phantom{000000} \\
 \underline{-28} \phantom{000000} \\
 20 \phantom{000000} \\
 \underline{-14} \phantom{000000} \\
 60 \phantom{000000} \\
 \underline{-56} \phantom{000000} \\
 40 \phantom{000000} \\
 \underline{-35} \phantom{000000} \\
 50 \phantom{000000} \\
 \underline{49} \phantom{000000} \\
 1 \phantom{000000}
 \end{array}$$

$$\begin{array}{r}
 \overline{0.285714} \\
 7 \sqrt{2.0} \\
 \underline{60} \\
 40 \\
 \underline{50} \\
 10 \\
 \underline{30} \\
 20
 \end{array}$$

$$\begin{array}{r}
 \overline{.428571} \\
 7 \sqrt{3.0} \\
 \underline{-2.8} \\
 20 \\
 \underline{-14} \\
 60 \\
 \underline{-56} \\
 40 \\
 \underline{-35} \\
 50 \\
 \underline{-49} \\
 10 \\
 \underline{-7} \\
 30
 \end{array}$$

$$\begin{array}{r}
 \overline{.571428571} \\
 7 \sqrt{4.000000} \\
 \underline{35} \\
 50 \\
 \underline{49} \\
 10 \\
 \underline{7} \\
 30 \\
 \underline{28} \\
 20 \\
 \underline{14} \\
 60 \\
 \underline{56} \\
 40 \\
 \underline{35} \\
 50 \\
 \underline{49} \\
 10
 \end{array}$$

$$\begin{array}{r}
 \overline{0.7142857142857} \\
 7 \sqrt{5.00000000} \\
 \underline{49} \\
 10 \\
 \underline{7} \\
 30 \\
 \underline{28} \\
 20 \\
 \underline{14} \\
 60 \\
 \underline{56} \\
 40 \\
 \underline{35} \\
 50 \\
 \underline{49} \\
 10 \\
 \underline{7} \\
 30 \\
 \underline{28} \\
 20
 \end{array}$$

$$\begin{array}{r}
 \overline{0.857142} \\
 7 \sqrt{6.0} \\
 \underline{56} \\
 40 \\
 \underline{35} \\
 50 \\
 \underline{49} \\
 10 \\
 \underline{7} \\
 30 \\
 \underline{28} \\
 20 \\
 \underline{14} \\
 60 \\
 \underline{56} \\
 40
 \end{array}$$

$$\begin{array}{r}
 \overline{.0101} \\
 11 \sqrt{1.0000} \\
 \underline{-0} \\
 100 \\
 \underline{-99} \\
 10 \\
 \underline{-10} \\
 0
 \end{array}$$

$$\begin{array}{r}
 \overline{.18} \\
 11 \sqrt{2.00} \\
 \underline{-11} \\
 90 \\
 \underline{88} \\
 20
 \end{array}$$

Notice that we have repeating decimals made up of the same digits. When you do  $1/7$  you're going through every possible remainder and shifting the decimal result according to the first remainder.

$$1/11 = .0909$$

$$2/11 = .1818$$

There are only two possible remainders when going through long division in either case so the repeating decimal has a repeating block of two digits.

All rational numbers will repeat!

Observe the rational number  $5/13$ . Because it can be written in the form  $p/q$ , where  $q$  does not equal zero, we know this will repeat eventually

$$5/13 = 0.3846153$$

$$6/13 = 0.461538$$

When we increase the numerator and keep the same denominator, the repeating decimal will shift over beginning with that remainder

$$\phi(14) = \{1, \cancel{2}, 3, \cancel{4}, 5, \cancel{6}, \cancel{7}, \cancel{8}, 9, \cancel{10}, 11, \cancel{12}, 13\}$$

$$\phi(27) = \phi(2) \cdot \phi(7) = 6$$

$$\begin{array}{cc} 11 & 11 \\ 1 & 6 \end{array}$$

$$\phi(27) = \phi(3^3) = 3^3 - 3^2 = 18$$

Euler's theorem to find the length of a repeating block

If we have a number  $n$

$\phi(n)$  = number of remainders that are relatively prime to  $n$

These will satisfy  $\gcd(r,n)=1$

The number of relatively prime numbers will be the repeating block length

$\phi(14)=\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$ , here there are 6 numbers relatively prime to 14 so the repeating block of a fraction with denominator 14 will be 6 digits long

Fermat's little theorem says the length of the repeating block will be  $10^{p-1}$  when  $p$  is a prime

When  $p=7$ , repeat after 6 places

$p=11$ , repeat after 10 places

$$\begin{array}{r}
 0.3846153 \\
 13 \overline{) 500000 \dots} \\
 \underline{39} \phantom{0000} \\
 110 \phantom{000} \\
 \underline{104} \phantom{00} \\
 60 \phantom{0} \\
 \underline{52} \\
 80 \\
 \underline{78} \\
 20 \\
 \underline{13} \\
 70 \\
 \underline{65} \\
 50
 \end{array}$$

$$10 \equiv_{13} -3 = 10$$

$$10^2 \equiv 9 \equiv (-3)^2$$

$$10^3 \equiv 9(-3) = -27 \equiv_{13} -1 \equiv_{13} 12$$

$$10^4 \equiv_{13} 9^2 = 3$$

$$10^5 \equiv_{13} 3 \cdot 3 = -9 \equiv_{13} 4$$

$$10^6 \equiv -9 - 3 \equiv_{13} 1$$

$$10^7 \equiv_{13} 10$$

$$\begin{array}{r}
 0.769230 \\
 13 \overline{) 100} \\
 \underline{91} \\
 90 \\
 \underline{78} \\
 120 \\
 \underline{117} \\
 30 \\
 \underline{26} \\
 40 \\
 \underline{39} \\
 0
 \end{array}$$

$$100 = 7 \cdot 13 + 9$$

$$1000 = 76 \cdot 13 + 12$$

ect.