Inverting Square Matrices

Methods for Solving Inverse Matrices

How can we find the inverse of a square matrix we are given? By combining our given matrix and the identity matrix into an augmented matrix. Here is an example of these steps given matrix A:

Why does this process give us A-1? We can think of the A-1 as a reciprocal of the original matrix A.

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} -1 & 2 & 1 & 0 \\ 3 & -4 & 0 & 1 \end{array}\right]$$

$$3R_1 + R_2 \to R_2$$

$$\left[\begin{array}{cc|cc} -1 & 2 & 1 & 0 \\ 0 & 2 & 3 & 1 \end{array}\right]$$

$$-R_1 \to R_1$$

$$\left[\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 0 & 2 & 3 & 1 \end{array}\right]$$

$$R_2 + R_1 \to R_1$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 2 & 3 & 1 \end{array}\right]$$

$$\tfrac{1}{2}R_2 \to R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1 & 3/2 & 1/2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 2 & 1 \\ 3/2 & 1/2 \end{bmatrix}$$

But what are we doing when we move from our first matrix [A| I] to the next matrix? We are technically transforming our 2x4 matrix [A| I] with another 2x2 matrix. Let us go back and rewrite all the steps as the multiplication of matrices.

$3R_1 + R_2 \to R_2$

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \left[\begin{array}{cc|cc} -1 & 2 & 1 & 0 \\ 3 & -4 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} -1 & 2 & 1 & 0 \\ 0 & 2 & 3 & 1 \end{array}\right]$$

$R_1 - R_2 \to R_1$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \left[\begin{array}{cc|cc} -1 & 2 & 1 & 0 \\ 0 & 2 & 3 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} -1 & 0 & -2 & -1 \\ 0 & 2 & 3 & 1 \end{array}\right]$$

$-R_1 \to R_1$ and $\frac{1}{2} R_2 \to R_2$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1/2 \end{bmatrix} \left[\begin{array}{cc|cc} -1 & 0 & -2 & -1 \\ 0 & 2 & 3 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1 & 3/2 & 1/2 \end{array}\right]$$

Now, looking back at our initial matrix A, if we multiply it by each of the transformation matrices on the left, that leaves us with the identity matrix. And, if we multiply the identity matrix by each of the transformation matrices, we obtain A-1. And now we are left with two equations of multiplication of matrices. We can remove the identity matrix from our second equation, since it will not change the final product of the other three matrices. Now, we find that A-1 is equal to the three transformation matrices. And, substituting A-1 for the three matrices in our top equation, we find that A-1 multiplied by A-1 gives us the identity matrix.

1.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3/2 & 1/2 \end{bmatrix}$$

Through this process, we are able to note that our inverse matrix is just a product of transformation matrices that returns A-1 to the identity matrix. This is the most basic way to find an inverse matrix, and does not require knowledge of the discriminant. Using the discriminant though, we are able to use Cramer’s Rule for finding the inverse matrix.

The determinant is a factor by which areas are scaled by a matrix, and it can be negative or positive. Determinants can help us determine whether or not a matrix will have an inverse matrix, and are needed in the formula for Cramer’s Rule. Let us go through an example of Cramer’s Rule below. First, we need to find the determinant.

$$A = \begin{bmatrix} 2 & 6 & 3 \\ 4 & -1 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

$$\begin{aligned} \det A &= 2\det\begin{bmatrix} -1 & 3 \\ 3 & 2 \end{bmatrix} + 6\det\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} + 3\det\begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix} \\ &= 2(-2-9) + 6(8-3) + 3(12+1) \\ &= -22 - 30 + 39 \\ &= -52 + 39 \\ &= -13 \end{aligned}$$

$$\det A = -13$$

So determinant of A=-13. Next, we solve for our Adj(A). Then, we take Adj(A) and transpose it. Now, we have almost found our A-1. Finally, we want to multiply the matrix by the reciprocal of the determinant, which leaves us with our final inverse matrix of the original matrix A.

$$A^{-1} = (\mathrm{Adj}\, A)^T = \frac{1}{\det A}\begin{bmatrix} -11 & -5 & 13 \\ -3 & 1 & 0 \\ 21 & 6 & -26 \end{bmatrix}$$

$$A^{-1} = -\frac{1}{13}\begin{bmatrix} -11 & -5 & 13 \\ -3 & 1 & 0 \\ 21 & 6 & -26 \end{bmatrix}$$

On a 2x2 matrix, Cramer’s Rule is a lot faster. Using this same formula, let us find out what A-1 would be for a generic 2x2 matrix A.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$Adj(A) = \begin{bmatrix} D & -C \\ -B & A \end{bmatrix}$$

$$(Adj\, A)^T = \begin{bmatrix} D & -B \\ -C & A \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} D & -B \\ -C & A \end{bmatrix}$$

Real Life Application

A group took a trip on a bus at $3 per child, and $3.20 per adult, for a total of $118.40. The same group takes a train back home, priced at $3.50 per child and $3.60 per adult for a total of $135.20. How many adults and children were in the group?

To start, we have two equations that we know. 3x+3.2y = 118.4, and 3.5x +3.6y=135.2. We can write this as multiplication of a matrix and a vector, to result in a vector.

$$\begin{array}{l} \text{bus} \\ \text{train} \end{array} \begin{bmatrix} 3 & 3.2 \\ 3.5 & 3.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 118.40 \\ 135.20 \end{bmatrix}$$

Now, we cannot divide matrices, so instead we need to take the inverse. What we do to one side will have to be done to the other, which leaves us with A-1Ax=A-1b. Now that we know that we need A-1, how do we find it? Below, we will solve for A-1 by using Cramer's Rule.

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ c & a \end{bmatrix}$$

$$= \frac{1}{10.8-11.2}\begin{bmatrix} 3.6 & -3.2 \\ -3.5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 8 \\ 8.75 & -7.5 \end{bmatrix}$$

$$\text{So} \quad \vec{x} = \begin{bmatrix} -9 & 8 \\ 8.75 & -7.5 \end{bmatrix}\begin{bmatrix} 118.4 \\ 135.2 \end{bmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -9(118.4) + 8(135.2) \\ 8.75(118.4) - 7.5(135.2) \end{pmatrix}$$

$$= \begin{pmatrix} 16 \\ 22 \end{pmatrix}$$

This leaves us with x=16, and y=22. So the group was made up of 16 children, and 22 adults.

Diagonal Matrices

A diagonal matrix is a square matrix whose diagonal entries are non-zero, and whose non-diagonal entries are zero. Finding the inverse matrix is simple, and we can write a formula for finding it.

We can prove this by checking if it will result in the identity matrix.

$$A = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix} \qquad A^{-1} \begin{bmatrix} \frac{1}{a} & 0 & 0 & 0 \\ 0 & 1/b & 0 & 0 \\ 0 & 0 & 1/c & 0 \\ 0 & 0 & 0 & 1/d \end{bmatrix}$$

Finding the Determinant

There is another trick to finding the determinant, that only works for 3x3 matrices. To find it, you create a 3x5 matrix by adding the first two of the matrix after the last three rows. Then, we look at the products of the diagonal lines. Those that end on the left of the middle column are multiplied by negative one, while those that end on the right are multiplied by one. Once the totals of the diagonals are found, we add them together and end up with the determinant

$$\begin{bmatrix} 2 & 6 & 3 \\ 4 & -1 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 6 & 3 & 2 & 6 \\ 4 & -1 & 3 & 4 & -1 \\ 1 & 3 & 2 & 1 & 3 \end{bmatrix}$$

$$-(-3) \quad \cancel{-18} \quad -48 \quad -4 \quad \cancel{18} \quad 36$$

$$= -45 - 4 + 36 = \boxed{-13}$$