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Permutation scribe notes:

Definitions:

Permutation: one of many ways a set or number of things can be ordered or arranged

Permutations:

Of {1, 2}

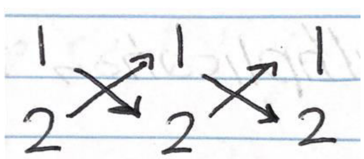
- Number of permutations is $2! = 2$

Of {1, 2, 3}

- Number of permutations is $3! = 6$

{1,2}: Permutations are a reflection. Call reflection r.

ID	other
1 -> 1	1 -> 2
2 -> 2	2 -> 1



So $r \circ r = Identity$

Identity

$1 \to 1$

$2 \to 2$

However, we can have other permutations:

$r(1) = 2$

$r(2) = 1$

IN TWO DIMENSIONS

We want to convert these to matrices

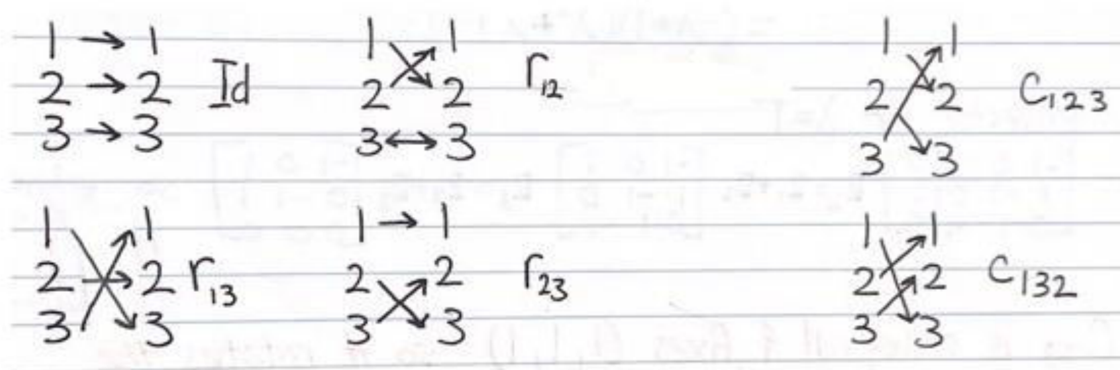
This gives us the Identity = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

The reflection matrix from this = $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

In fact, squaring the reflection matrix gives back the identity! $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

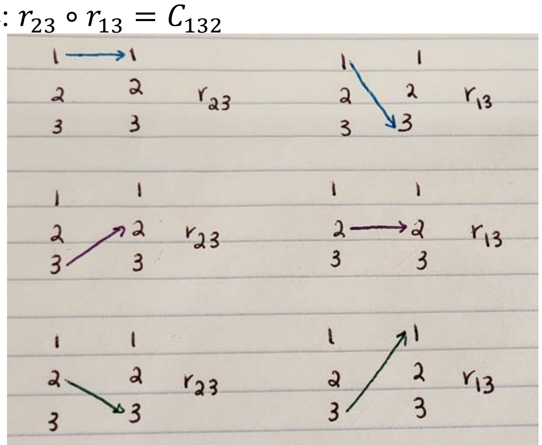
IN THREE DIMENSIONS

- We can have more permutations on which number goes to which. In fact, we can see that there are 6 cases that we choose from. (All of which are shown in the image below)
    - Identity
    - r_{12}
    - r_{13}
    - r_{23}
    - C_{123}
    - C_{132}



We also see that doing the dot product for any two of these permutations gives you a different permutation on the list.

- For example: $r_{23} \circ r_{13} = C_{132}$



We Can Create a Group Permutation Multiplication Table in 3-Dimensions

	I	(1 2)	(1 3)	(2 3)	(1 2 3)	(1 3 2)
I	I	(1 2)	(1 3)	(2 3)	(1 2 3)	(1 3 2)
(1 2)	(1 2)	I	(1 2 3)	(1 3 2)	(1 3)	(2 3)
(1 3)	(1 3)	(1 3 2)	I	(1 2 3)	(2 3)	(1 2)
(2 3)	(2 3)	(1 2 3)	(1 3 2)	I	(1 2)	(1 3)
(1 2 3)	(1 2 3)	(2 3)	(1 2)	(1 3)	(1 3 2)	I
(1 3 2)	(1 3 2)	(1 3)	(2 3)	(1 2)	I	(1 2 3)

Converting Permutation Groups into Matrices:

Identity $\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$r_{12} \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$C_{123} \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Quick Note: Permutation Matrices are ALL orthogonal:

- Meaning that $C_{123} \cdot {C_{123}}^T = Identity$
  - ${C_{123}}^T = {C_{123}}^{-1} = C_{132}$

Cyclic Notation:

In mathematics, and in particular in group theory, a cyclic permutation (or cycle) is a permutation of the elements of some set X which maps the elements of some subset S of X to each other in a cyclic fashion, while fixing (that is, mapping to themselves) all other elements of X. If S has k elements, the cycle is called a k-cycle. Cycles are often denoted by the list of their elements enclosed with parentheses, in the order to which they are permuted.

For example, given X = {1, 2, 3, 4}, the permutation (1, 3, 2, 4) that sends 1 to 3, 3 to 2, 2 to 4 and 4 to 1 (so S = X) is a 4-cycle, and the permutation (1, 3, 2) that sends 1 to 3, 3 to 2, 2 to 1 and 4 to 4 (so S = {1, 2, 3} and 4 is a fixed element) is a 3-cycle. On the other hand, the permutation that sends 1 to 3, 3 to 1, 2 to 4 and 4 to 2 is not a cyclic permutation because it separately permutes the pairs {1, 3} and {2, 4}.

Writing our permutations into cyclic notation:

- $r_{12} = (1\ 2)(3) = (1\ 2)$
- $C_{123} = (1\ 2\ 3)$

Looking at C_{123}:

$$\det\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = 1$$

Finding the Characteristic Polynomial/Eigenvalues:

$$det[C_{123} - \lambda I] = det\begin{bmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix} = -\lambda(\lambda^2) + 1 = -\lambda^3 + 1$$

$$= (-\lambda + 1)(\lambda^2 + \lambda + 1)$$

$\lambda = 1$ Which has a corresponding eigenvector of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

This means that C_{123} is orthogonal and fixes (1, 1, 1). So it rotates the plane orthogonal to (1, 1, 1)

In fact: Every permutation fix (1, 1, 1) and all of them are orthogonal transformations of $x + y + z = 0$

Finding the simplest possible set of matrices:

- We can do this with 2x2 matrices

$$\left\{ \begin{array}{ll} id \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & r_{23} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \\ r_{12} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} & C_{123} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \\ r_{13} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} & C_{132} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \end{array} \right\}$$

History:

The study of groups originally grew out of an understanding of permutation groups. Al-Khalil, an Arab mathematician wrote about permutations in his book *Book of Cryptographic* Messages in order to list all possible Arabic words with and without vowels. The rule to determine the number of permutations of n objects came out of India through the mathematician Bhaskara II. Permutations had themselves been intensively studied by Lagrange in 1770 in his work on the algebraic solutions of polynomial equations. The subject flourished and by the mid 19th century a well-developed theory of permutation groups extended.

Curriculum:

In Secondary Mathematics III, we see the application of permutations in Standard S.CP.9 where students go into depth on using and finding permutations and combinations to compute probabilities of compound events and solve problems.

However, permutation matrices come into play in Linear Algebra in college.

Applications:

- Central to the study of symmetries and to Galois Theory
- Computer Science
- Quantum Physics
- Encryption Process