Mike
Rebekah
Cassandra

Characteristic Polynomials and Eigenvalues of Square Matrices (11/7)

λ is an **eigenvalue** of an nxn matrix, A, if the following holds:
$Ax = \lambda x \quad \rightarrow \quad (A - \lambda I)x = 0$

Let us try to find the eigenvalues of the matrix $\begin{vmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{vmatrix}$.

det$\begin{vmatrix} -1-\lambda & \sqrt{3} \\ \sqrt{3} & -3-\lambda \end{vmatrix} = (-1-\lambda)(-3-\lambda) - 3$
$= 3 + 3\lambda + \lambda + \lambda^2 - 3$
$= \lambda^2 + 4\lambda$
$= \lambda(4 + \lambda)$

So the eigenvalues are $\lambda = 0, -4$
To find our eigenvectors, we will plug in our eigenvalues to the augmented matrix for $(A - \lambda I)x = 0$. So for $\lambda = 0$,

$$\begin{vmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{vmatrix} \begin{matrix} 0 \\ 0 \end{matrix}$$
$$\begin{vmatrix} -1 & \sqrt{3} \\ 0 & 0 \end{vmatrix} \begin{matrix} 0 \\ 0 \end{matrix}$$

So $x = \begin{vmatrix} \sqrt{3} \\ 1 \end{vmatrix}$.

For $\lambda = -4$,

$$\begin{vmatrix} -1+4 & \sqrt{3} \\ \sqrt{3} & -3+4 \end{vmatrix} \begin{matrix} 0 \\ 0 \end{matrix}$$
$$\begin{vmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{vmatrix} \begin{matrix} 0 \\ 0 \end{matrix}$$

So $x = \begin{vmatrix} 1 \\ -\sqrt{3} \end{vmatrix}$.

<u>Reflection y=x</u>

$$A = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} x \\ y \end{vmatrix}$

$det(A - \lambda I) = det\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1) = 0$
So, $\lambda = 1, -1$

$\lambda = -1$:

$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix} \quad \rightarrow \quad \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad \rightarrow \quad x = -y \rightarrow \begin{vmatrix} -1 \\ 1 \end{vmatrix}$

$\lambda = 1$:

$\begin{vmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} \quad \rightarrow \quad \begin{vmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad \rightarrow \quad \begin{vmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$

x=y → $\begin{vmatrix} 1 \\ 1 \end{vmatrix}$

y=y

Check:

$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\begin{vmatrix} 1 \\ 1 \end{vmatrix} = \begin{vmatrix} 1 \\ 1 \end{vmatrix} \quad \rightarrow \quad 1\begin{vmatrix} 1 \\ 1 \end{vmatrix} = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$

<u>Reflection across the line y=tan(θ/2)</u>

A= $\begin{vmatrix} cos\theta & sin\theta \\ sin\theta & -cos\theta \end{vmatrix}$

$$det(A - \lambda I) = det\begin{vmatrix} cos\theta-\lambda & sin\theta \\ sin\theta & -cos\theta-\lambda \end{vmatrix}$$

= $-(cos^2(\theta) - \lambda^2) - sin^2(\theta)$ = $\lambda^2 - 1$ → λ =1,-1

λ =1:

$\begin{vmatrix} cos\theta-1 & sin\theta & 0 \\ sin\theta & -cos\theta-1 & 0 \end{vmatrix}$ → Bottom row goes to 0

= $(cos(\theta) - 1)x + sin(\theta)y = 0$ → $(cos(\theta) - 1)x = -sin(\theta)y$

→ x=1 then $y = \frac{1-cos\theta}{sin\theta} = tan(\theta/2)$

<u>Rotation by θ</u>

When we rotate by θ, we are rotating all vectors in the plane. So, we need our matrix to rotate all vectors in the plane. We can find that our 2x2 rotation matrix A will equal $\begin{vmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{vmatrix}$. In order to find the eigenvalues of matrix, we can take the determinant of $(A - \lambda I)$.

$$\det(A - \lambda I) = \begin{vmatrix} cos\theta-\lambda & -sin\theta \\ sin\theta & cos\theta-\lambda \end{vmatrix} = cos^2\theta - 2cos\theta + \lambda^2 + sin^2\theta$$
$$= 1 - 2cos\theta + \lambda^2$$

$$\lambda = \cos\theta \pm \sqrt{\cos^2\lambda - 1}$$

$$\lambda = \cos\theta \pm \sin\theta\, i$$
$$\lambda = e^{i\theta}, e^{-i\theta}$$

In this example, we see that we are C^2, which is a 4 dimensional space. Now that we have the eigenvalues of our matrix, we want to find the eigenvectors. We'll do the same thing we did earlier, and solve our augmented matrix to find $(A - \lambda I)x = 0$.

Let us start with $\lambda = e^{i\theta}$:

$$\begin{vmatrix} cos\theta-e^{i\theta} & -sin\theta \\ sin\theta & cos\theta-e^{i\theta} \end{vmatrix} \begin{matrix} 0 \\ 0 \end{matrix}$$
$$\begin{vmatrix} cos\theta-e^{i\theta} & -sin\theta \\ 0 & 0 \end{vmatrix} \begin{matrix} 0 \\ 0 \end{matrix}$$
$$\begin{vmatrix} -isin\theta & -sin\theta \\ 0 & 0 \end{vmatrix} \begin{matrix} 0 \\ 0 \end{matrix}$$
$$\begin{vmatrix} i & 1 \\ 0 & 0 \end{vmatrix} \begin{matrix} 0 \\ 0 \end{matrix}$$

So $x_1 = 1;\ x_2 = -i$. So $x = \begin{vmatrix} 1 \\ -i \end{vmatrix}$.

Let us know work with $\lambda = e^{-i\theta}$:

$$\begin{vmatrix} cos\theta-e^{-i\theta} & -sin\theta \\ sin\theta & cos\theta-e^{-i\theta} \end{vmatrix} \begin{matrix} 0 \\ 0 \end{matrix}$$
$$\begin{vmatrix} cos\theta-e^{-i\theta} & -sin\theta \\ 0 & 0 \end{vmatrix} \begin{matrix} 0 \\ 0 \end{matrix}$$
$$\begin{vmatrix} isin\theta & -sin\theta \\ 0 & 0 \end{vmatrix} \begin{matrix} 0 \\ 0 \end{matrix}$$
$$\begin{vmatrix} i & -1 \\ 0 & 0 \end{vmatrix} \begin{matrix} 0 \\ 0 \end{matrix}$$

So we find that $x = \begin{vmatrix} 1 \\ i \end{vmatrix}$.

Orthogonal Matrices

An orthogonal matrix A in R^3 is one such that $AA^T = I_3$, and the lengths of A are preserved. This means that $||Av|| = ||v||$. So for an orthogonal matrix, if $Av = \lambda v$, then $||Av|| = |\lambda|||v||$. So if λ is real, then $|\lambda| = 1$ and $\lambda = \pm 1$. If λ is complex, then $\lambda = cos\theta \pm isin\theta$. This orthogonal matrix is orthogonal in 3 dimensions.

If we take $det(A - \lambda I)$, we will get back a polynomial of degree three. Every polynomial of degree three must cross the x axis somewhere, so we will have a real solution of ± 1.