How to Multiply Complex Numbers in $re^{i\theta}$ Form and Extracting n^{th} Roots

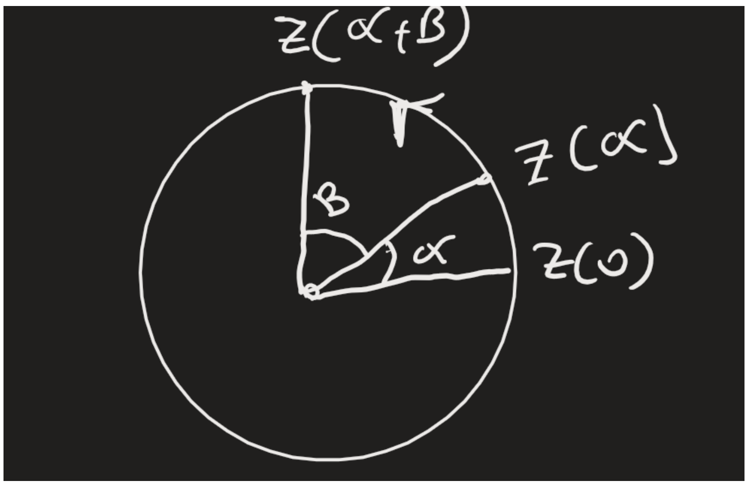
Statement:

Multiplication of complex numbers is a process of the multiplication of two or more complex numbers using the distributive property.

Prove $e^{i\theta} = cos\,(\theta)\ + i * sin\,(\theta)$

First let,

$$z(\theta) = cos\,(\theta)\ + i * sin\,(\theta)$$



Then take the derivative of z,

$$z'(\theta) = -sin\,(\theta)\ + i * cos\,(\theta)$$

Simplifying,

$$z'(\theta) = i^2 * sin\,(\theta)\ + i * cos\,(\theta)$$

$$z'(\theta) = i * [i * sin\,(\theta)\ + cos\,(\theta)\,]$$

$$z'(\theta) = i * z(\theta)$$

$$\frac{z\prime(\theta)}{z(\theta)} = i$$

Take the integral of both sides,

$$\int \quad \frac{z'(\theta)}{z(\theta)} = \int \quad i\, d\theta$$

Then we get,

$$ln\,\big(z(\theta)\big)\ = i\theta + C$$

Simplifying,

$$e^{ln\,(z(\theta))} = e^{i\theta + C}$$

$$z(\theta) = e^{i\theta} e^{C} = Ce^{i\theta}$$

$$cos\,(\theta)\; + isin(\theta) = Ce^{i\theta}$$

When $\theta = 0$,

$$1 + 0 = C(1)$$

$$C = 1$$

Simplifying,

$$e^{i\theta} = cos\,(\theta)\; + i * sin\,(\theta)$$

Angular Addition Formulas:

1. $cos\,(\alpha + \beta)\; = cos\,(\alpha)\,cos(\beta) - sin\,(\alpha)\,sin\,(\beta)$
2. $sin\,(\alpha + \beta)\; = sin\,(\alpha)\,sin\,(\beta)\; + cos\,(\alpha)\,sin\,(\beta)$

Check:

$$e^{i(\alpha+\beta)} = e^{i\alpha} * e^{i\beta}$$

We know that

$$e^{i\theta} = cos\,(\theta)\; + i * sin\,(\theta)$$

Therefore,

$$e^{i(\alpha+\beta)} = cos\,(\alpha + \beta)\; + i * sin\,(\alpha + \beta)$$

We know that

$$cos\,(\alpha + \beta)\; = cos\,(\alpha)\,cos(\beta) - sin\,(\alpha)\,sin\,(\beta)$$

$$sin\,(\alpha + \beta)\; = sin\;sin\,(\alpha)\,cos\,(\beta)\; + cos\,(\alpha)\,sin\,(\beta)$$

Therefore, we can simplify to

$$e^{i\alpha} * e^{i\beta} = [cos\,(\alpha)\; + i * sin\,(\alpha)\,] * [cos\,(\beta)\; + i * sin\,(\beta)\,]$$

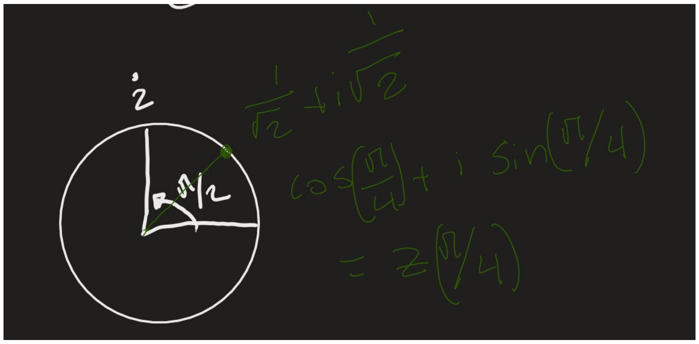
$$e^{i\alpha} * e^{i\beta} = cos\,(\alpha)\,cos\,(\beta)\; - sin\,(\alpha)\,sin\,(\beta)\; + i * [sin\,(\alpha)\,cos\,(\beta) + cos\,(\alpha)\,sin\,(\beta)\,]$$

$$e^{i\alpha} * e^{i\beta} = cos\,(\alpha + \beta)\; + i * sin\,(\alpha + \beta)$$

We have proved that

$$e^{i(\alpha+\beta)} = e^{i\alpha} * e^{i\beta}$$

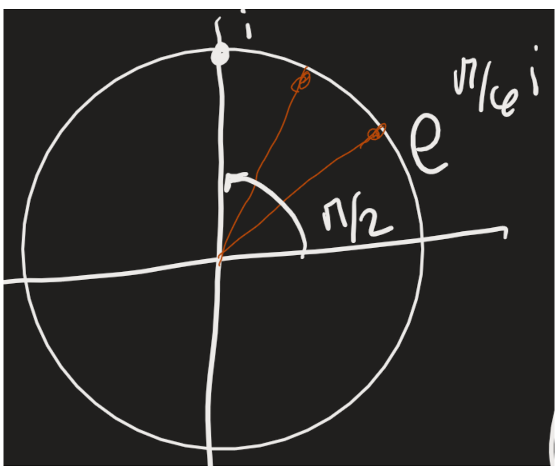
Square Roots of $i = e^{\frac{\pi}{2}i}$:

1. $e^{\frac{5\pi}{4}i}$ *and* $e^{\frac{\pi}{4}i}$
2. 

Cube Roots of $i = e^{\frac{\pi}{2}i}$:

1. $(e^{\frac{\pi}{6}i})^3$

$$e^{\frac{\pi}{6}i} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$



Using Binomial Theorem:

$$\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^3 = \left(\frac{\sqrt{3}}{2}\right)^3 + 3\left(\frac{\sqrt{3}}{2}\right)^2\left(\frac{1}{2}i\right) + 3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}i\right)^2 = \frac{3\sqrt{3}}{8} + \frac{9}{8}i + \frac{3\sqrt{3}}{8}(-1) + \frac{1}{8}(-i)$$
$$= i$$

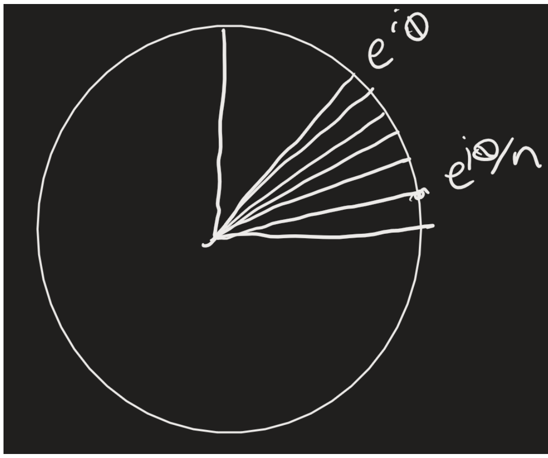
What if we want to find the $\sqrt{2i} = \sqrt{2}\sqrt{i}$

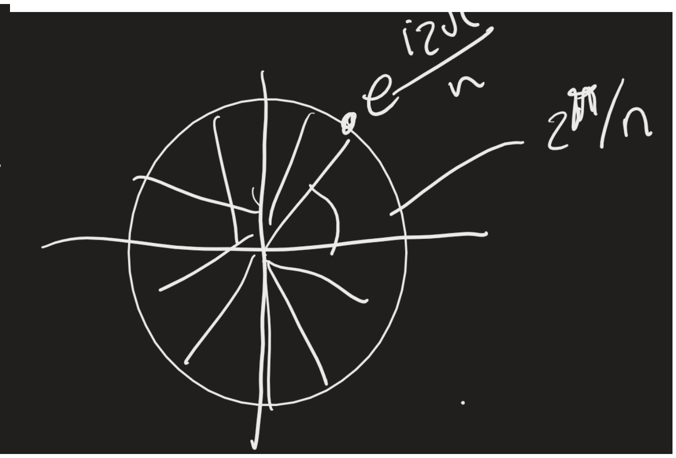
$$= \sqrt{2}e^{(\pi/4)i} \text{ or } \sqrt{2}e^{(5\pi/4)i}$$

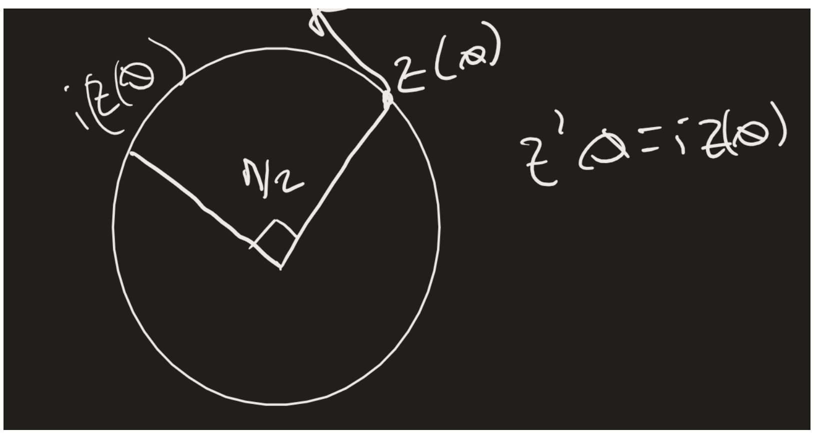
$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \text{ or } \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = (1 + i) \text{ or } (-1 - i)$$

Nth Roots:

$$\sqrt[n]{e^{i\theta}} = e^{\frac{i\theta}{n}}, e^{\frac{i(\theta+2\pi)}{n}}, \ldots$$







Polar Form in Exponential Form

$$z = r(cos\theta + isin\theta) \rightarrow z = re^{i\theta}$$

History:

The square root of negative numbers created a conundrum that led mathematician Gerolamo Cardano to conceive up complex numbers in around 1545 in his Ars Magna. Many mathematicians contributed to the development of complex numbers as the rules for addition, subtraction, multiplication and root extraction were developed by Rafael Bombelli.

Applications:

- Signal Processing
- Control Theory
- Electromagnetism

- Quantum Mechanics
- Cartography

Curriculum:

This specific subject is in the curriculum in Secondary Mathematics III in high schools in the honors standards of N.CN.3-N.CN.6 & N.CN.10