Math 4030
Binomial Theorem, Pascal's Triangle, Fermat's Little Theorem
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Definitions:

- **Binomial**
  - **An algebraic expression with two terms**
- **Rational Number**
  - **A number that can be expressed as a quotient or fraction p/q of two integers**
- **Pascal's Triangle**
  - **The further expansion to find the coefficients of the Binomial Theorem**

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Binomial Theorem
STATEMENT:

- The Binomial Theorem is a quick way of expanding a binomial expression that has been raised to some power. For example, $(3x - 2)$ is a binomial, if we raise it to an arbitrarily large exponent of 10, we can see that $(3x - 2)^{10}$ would be painful to multiply out by hand.

Formula for the Binomial Theorem:

$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ Where: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

N choose K is called so because there is (n/k) number of ways to choose k elements, irrespective of their order from a set of n elements.

Proof by Induction:
Noting $i = k + 1$
Basis Step: $n = 1$
$(a + b)^1 = a + b \rightarrow \sum_{k=0}^{1} \binom{1}{k} a^{1-k} b^k$

$$\frac{1!}{0!\,1!} a^1 b^0 + \frac{1!}{1!\,0!} a^0 b^1 = 1a + 1b$$

Induction Step:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$(a+b)^{n+1} = (a+b)(a+b)^n$$

$$= (a+b)\left[\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k\right]$$

$$= a\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k + b\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1}$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=0}^{n-1} \binom{n}{k} a^{n-k} b^{k+1}$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^k + b^{n+1} + \sum_{i=1}^{n-1} \binom{n}{i-1} a^{n-i+1} b^i$$

$$= a^{n+1} \sum_{k=0}^{n} \binom{n}{k} a^{n-k+1} b^k + b^{n+1} + \sum_{i=1}^{n-1} \binom{n}{i-1} a^{n-i+1} b^i$$

$$= a^{n+1} + b^{n+1} + \sum_{t=1}^{n} [\binom{n}{t} + \binom{n}{t-1}] a^{n-t+1} b^t$$

$$= a^{n+1} + b^{n+1} + \sum_{t=1}^{n} [\binom{n+1}{t} + a^{n+1-t} b^t] = \sum_{t=0}^{n} [\binom{n+1}{t} a^{n+1-t} b^t]$$

When n is true, $n+1$ is true; so the statement holds for all n ≥ 1

Question:
If you throw 4 coins: what is the possibility that it will land on 3 heads and 3 tails
Let a be heads and b be tails.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

1
2
4
8
16
32
64

If you want to know the probability that you will get 2 heads and 2 tails, looking at pascal's triangle, we see that it falls under the number 6 and so the probability would be 6 over the total number of possibilities on that row. Which added up on all the numbers is 16 possibilities. So the total probability would be $\frac{6}{16} = 37.5\%$

Example Using the Binomial Theorem:
Find $(\boldsymbol{a} + \boldsymbol{b})^3$

Using our formula we can see that $(a + b)^3$ goes out to ${}_3C_0a^3 + {}_3C_1a^2b + {}_3C_2ab^2 + {}_3C_3b$

$$= \left(\frac{3!}{(3-0)!\,0!}\right)a^3 + \left(\frac{3!}{(3-1)!\,1!}\right)a^2b + \left(\frac{3!}{(3-2)!\,2!}\right)ab^2 + \left(\frac{3!}{(3-3)!\,3!}\right)b^3$$

$$= (1)a^3 + (3)a^2b + (3)ab^2 + (1)b^3 = \boldsymbol{a^3 + 3a^2b + 3ab^2 + b^3}$$

<u>Another way of looking at Binomial Expansion</u>

$$(x + y)^5 = 1x^0y^5 + 5x^1y^4 + 10x^2y^3 + 10x^3y^2 + 5x^4y^1 + 1x^5y^0$$

Where you get the coefficients from the $(n + 1)$ row of Pascal's Triangle. Where n is the power you are raising to.

Statement of Fermat's Little Theorem:

- Let p be a prime which does not divide the integer k then $k^{p-1} \equiv 1 (mod\ p)$ or $k^{p-1} \equiv_p 1$

Prove:

Claim: $n^p \equiv_p n$.

Example:

$p = 5\ and\ n = \{0, 1, 2, 3, 4\}$

$$0^5 = 0\ mod\ 5$$

$$1^5 = 1\ mod\ 5$$

$$2^5 = 32 \equiv 2\ mod\ 5$$

$$3^5 = 243 \equiv 3\ mod\ 5$$

$$4^5 = 1024 \equiv 4\ mod\ 5$$

Induction Proof:

- Basic Step: n = 1

$1^p = 1 \equiv_p 1$

Induction Step:

- $(n+1)^p \equiv_p (n+1)$

$(n+1)^p = \sum_{k=0}^{p} \binom{p}{k} n^{p-k} 1^k \equiv_p 1n^p 1^0 + 1n^0 1^p = n^p + 1^p = n + 1$

So Fermat's Little Theorem is true by induction. ■

History:

The first formulation of the binomial theorem and the table of binomial coefficients can be found in a work by Al-Karaji in the 12th-13th century. He also provided the proof of both the Binomial Theorem and Pascal's Triangle.

Curriculum:

Students do not normally see the theorem for binomial expansion and how it relates back to the use of Pascal's Triangle until they reach the level of Secondary Math III. This is shown in the 11th grade standard of A.APR.5 which states: **Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers. For example, with coefficients determined by Pascal's Triangle.**