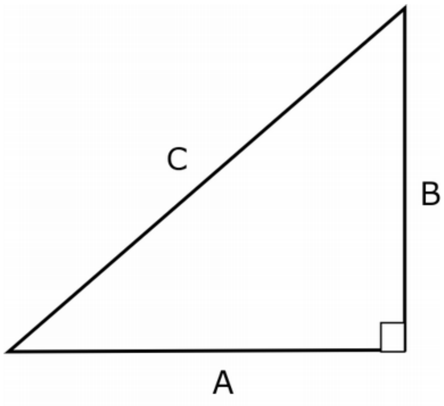
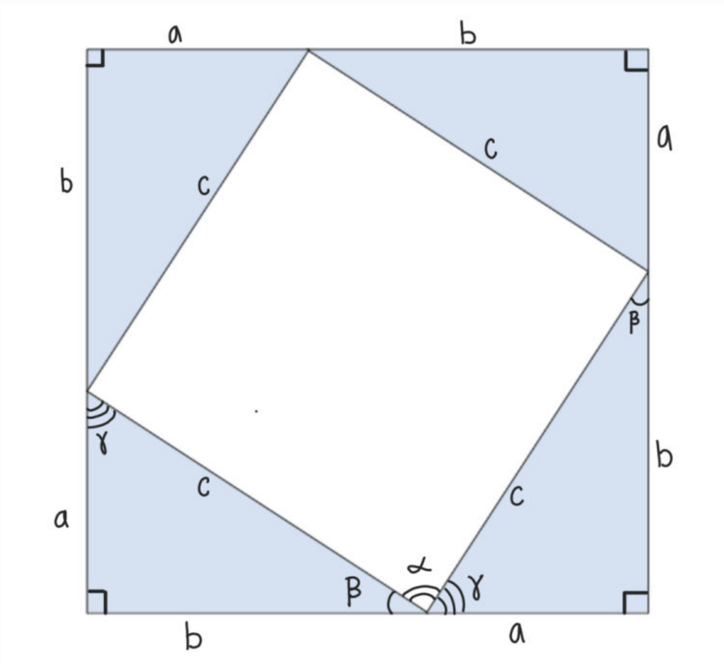
The Pythagorean Theorem and Other Trig Identities



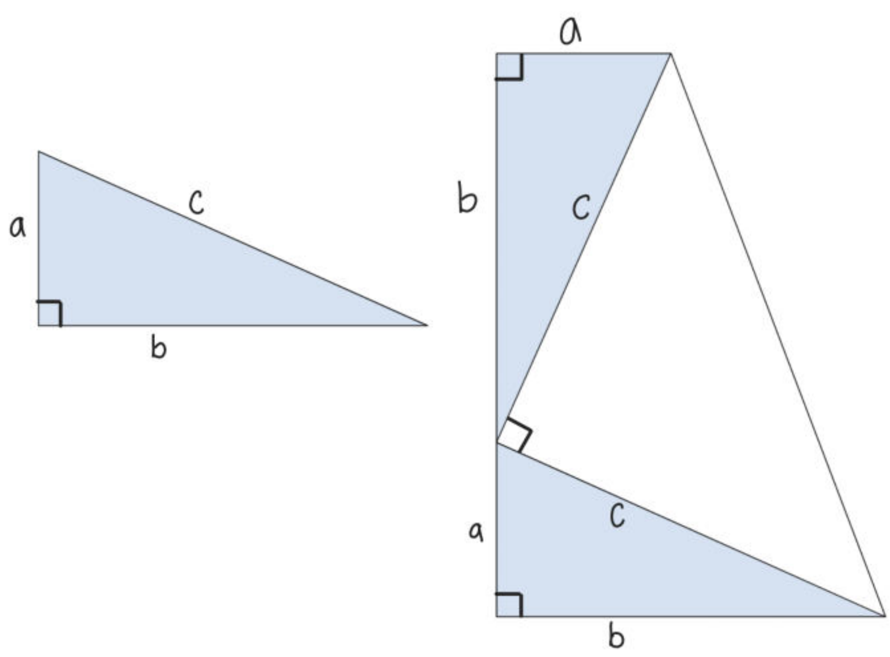
- The Pythagorean Theorem: Given a right triangle, we have the following relation: $a^2 + b^2 = c^2$ where a and b are two legs of the right triangle and c is the hypotenuse (the segment directly across from the $90°$ angle).
- The Pythagorean Theorem has several proofs, although each proof is mostly a rearrangement of the last.

1. Traditional Proof:



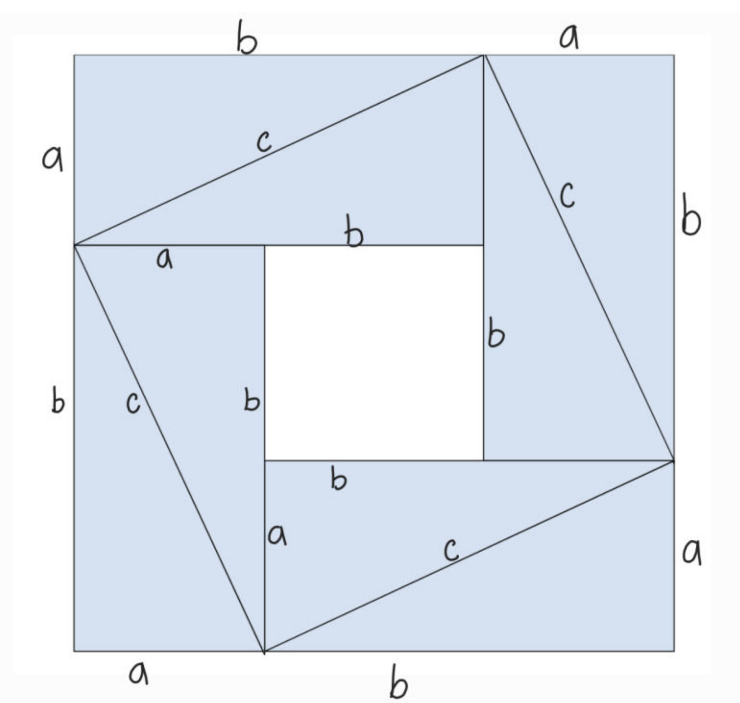
- In this proof, we have four congruent triangles placed together. We want to know if the inside figure is a square.
  - Angles $\beta + \alpha + \gamma = 180°$ since they form a straight line.
  - Also, $\beta + \gamma = 90°$ since they are the other two angles in a right triangle, they must add to $90°$.
- Area of the entire figure is $(a + b)(a + b) = a^2 + 2ab + b^2$
- Area of the inner square plus the four triangles: $c^2 + 4(\frac{1}{2}ab) = c^2 + 2ab$
- These two areas are equal:
  - $a^2 + 2ab + b^2 = c^2 + 2ab$
    - $a^2 + b^2 = c^2$

2. Garfield's Proof:



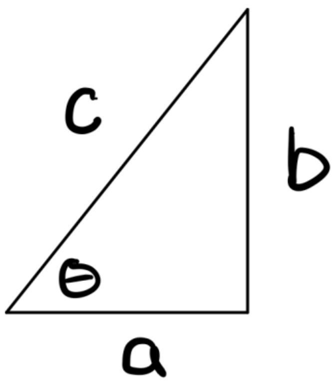
- Let the figure to the left be a trapezoid formed by two congruent triangles put together. Then the area of the trapezoid is $\frac{1}{2}(a+b)(a+b)$
- The area of the parts forming the trapezoid is $\frac{1}{2}c^2 + 2(\frac{1}{2})ab$
- Since these two areas are equal, we can say that
  - $\frac{1}{2}(a+b)(a+b) = \frac{1}{2}c^2 + 2(\frac{1}{2})ab$
  - $\frac{1}{2}a^2 + ab + \frac{1}{2}b^2 = \frac{1}{2}c^2 + ab$
  - $\frac{1}{2}a^2 + \frac{1}{2}b^2 = \frac{1}{2}c^2$
  - $a^2 + b^2 = c^2$

3. One final proof:



- The area of the middle tilted square is c^2.
- The area of the small white square is $(b-a)^2$.
- The area of the triangles surrounding the small white square is $4(\frac{1}{2})ab$.
- The area of the middle titled square is equal to the area of the white square plus the areas of the four triangles.
  - $c^2 = (b-a)^2 + 4(\frac{1}{2})ab$
  - $c^2 = b^2 - 2ab + a^2 + 2ab$
  - $c^2 = a^2 + b^2$

Trigonometry Identities



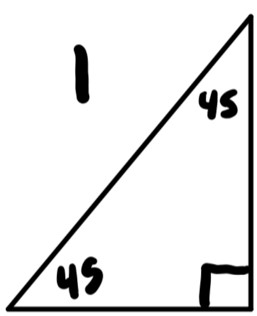
Given a triangle abc with sides shown below, we can find the value of specific trig functions:

$$sin\theta = \frac{b}{c} \qquad cos\theta = \frac{a}{c} \qquad tan\theta = \frac{b}{a}$$
$$csc\theta = \frac{c}{b} \qquad sec\theta = \frac{c}{a} \qquad cot\theta = \frac{a}{b}$$

We can use these identities to show that:

$$sin^2\theta + cos^2\theta = 1 \ \Rightarrow \frac{b^2}{c^2} + \frac{a^2}{c^2} = 1$$
$$\Rightarrow b^2 + a^2 = c^2$$
$$sec^2\theta - tan^2\theta = 1 \Rightarrow \frac{c^2}{a^2} - \frac{b^2}{a^2} = 1$$
$$\Rightarrow c^2 - b^2 = a^2$$
$$csc^2\theta - cot^2\theta = 1 \Rightarrow \frac{c^2}{b^2} - \frac{a^2}{b^2} = 1$$
$$\Rightarrow c^2 - a^2 = b^2$$

The Unit Circle and Triangles on it

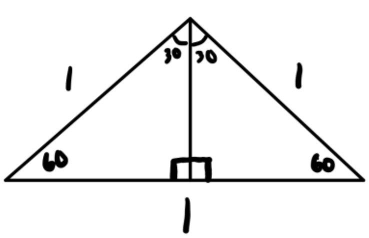


If we assume our hypotenuse is a radius of the unit circle, we know its length will be 1. Then, we know that using the Pythagorean theorem:

$$a^2 + b^2 = 1$$

Since both of our angles adjacent to the hypotenuse are the same, our two side lengths, a and b must be the same. So:

$$a^2 + a^2 = 1 \Rightarrow 2a^2 = 1 \Rightarrow a^2 = 1/2 \Rightarrow a = \frac{1}{\sqrt{2}}$$

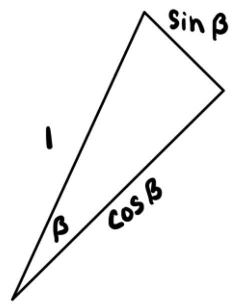


If we again let our hypotenuse by the radius of the unit circle, we know its length will be 1. And, if we were to reflect our triangle over the side opposite our 60^o, we find that we create an isosceles triangle, with three 60^o angles. As such, we know all of our sides must be 1. So the base of one triangle will be $\frac{1}{2}$. Then, using the Pythagorean theorem, we find our missing side length is equal to:

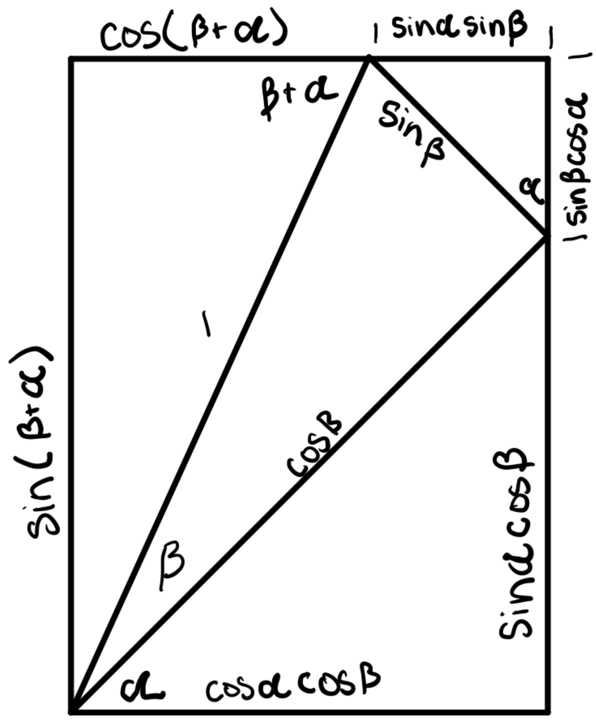
$$\sqrt{1^2 - (\frac{1}{2})^2} = \ \frac{\sqrt{3}}{2}$$

Proof that $sin(x \pm y) = sinxcosy \pm cosxsiny$

Let us say we have a triangle with one 90^o angle, one angle β and sides 1, $sin\beta$, and $cos\beta$, as drawn below:



Then by drawing a rectangle around the triangle, we find the following:



Since we have a rectangle, with four 90^o angles, we find the two following statements to be facts:

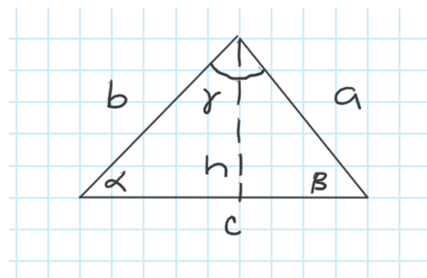
$$sin(\beta + \alpha) = sin(\beta)cos(\alpha) + sin(\alpha)cos(\beta)$$

$$cos(\alpha)cos(\beta) = cos(\beta + \alpha) + sin(\alpha)sin(\beta)$$

Law of Sines and Cosines

1. Law of Sines: $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$

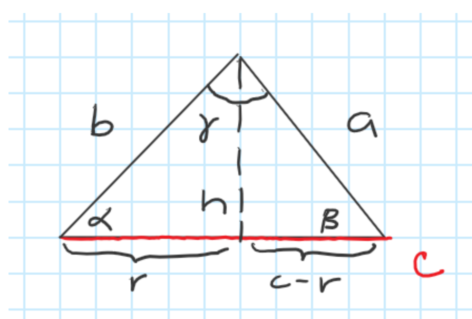
   Proof:



   Since $sin(x) = \frac{opposite}{hypotenuse}$, we can see that $sin(\alpha) = \frac{h}{b}$ and $sin(\beta) = \frac{h}{a}$.
   Solving for h in both equations, we get $h = bsin(\alpha)$ and $h = asin(\beta)$.
   Therefore, we can rewrite this as $bsin(\alpha) = asin(\beta)$
   Dividing both sides by b, we get $sin(\alpha) = \frac{asin(\beta)}{b}$.
   Dividing both sides by a, we get $\frac{sin(\alpha)}{a} = \frac{sin(\beta)}{b}$.
   We can use a similar argument to prove $\frac{sin(\beta)}{b} = \frac{sin(\gamma)}{c}$.

2. Law of Cosines: $a^2 = b^2 + c^2 - 2bc * cos(\alpha)$

   Proof:



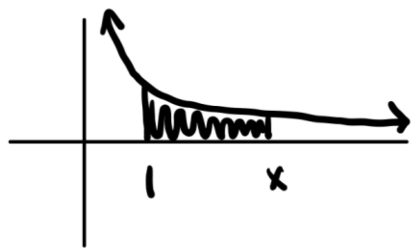
   Since $sin(x) = \frac{opposite}{hypotenuse}$, we can see that $sin(\alpha) = \frac{h}{b}$ and $cos(\alpha) = \frac{r}{b}$.
   Solving for h and r, respectively, we get $h = bsin(\alpha)$ and $r = bcos(\alpha)$.
   By Pythagorean Theorem, we can see that $a^2 = h^2 + (c - r)^2$.
   Using our values of h and r in this equation we get, $a^2 = (bsin(\alpha))^2 + (c - bcos(\alpha))^2$.
   Simplifying we get, $a^2 = b^2sin^2(\alpha) + c^2 + b^2cos^2(\alpha) - 2bc * cos(\alpha)$.

$$a^2 = b^2(sin^2(\alpha) + cos^2(\alpha)) + c^2 - 2bc * cos(\alpha)$$

$$a^2 = b^2 + c^2 - 2bc * cos(\alpha)$$

Logarithms

The logarithm function is defined as $ln(x) = \int_1^x \frac{1}{t} dt$. So $ln(t)$ will be the area under curve of the function $\frac{1}{x}$ starting from when $x = 1$ and ending at $x = t$. We can see this in the drawing below:



So we see that $ln(1) = 0$. And $ln(x) < 0$ if $0 < x < 1$. How might we solve $ln(xy) = \int_1^{xy} \frac{1}{t} dt$, where $x, y \geq 1$? Well, we can split up our definite integral into two different definite integrals. So $ln(xy) = \int_1^x \frac{1}{t} dt + \int_x^{xy} \frac{1}{t} dt$. Then ,we know what the value of $\int_1^x \frac{1}{t} dt$ is, as we can see in our drawing. And, if we substitute in $u = \frac{t}{x}$, Then we find $ln(xy) = ln(x) + \int_1^y \frac{1}{ux} d(ux) = ln(x) + \int_1^y \frac{1}{u} du = ln(x) + ln(y)$. And we can see this in the drawing below:

Though students are introduced to the function e^x before they see logarithms, we define e^x to be the inverse function of $ln(x)$.