Math 3210-4/ Honors Foundations of Analysis/Fall 2016
Third Midterm (Due Wednesday December 14)

1. Using the fact that the real numbers are an ordered field with the least upper bound property, and that rational numbers are real, prove that for every real number $\epsilon > 0$, there is a positive integer n so that

$$1/n < \epsilon$$

2. If $E \subset X$ is a subset of a metric space, then E is disconnected if there are subsets $A, B \subset E$ that are open relative to E such that:

$$A \cup B = E,\ \ A \cap B = \emptyset \text{ and } A, B \neq \emptyset$$

Otherwise E is connected. Show that this definition agrees with the definition in the book (2.45).

3. Let $\mathbb{R}^\infty$ be the set of sequences $\underline{a} = (a_1, a_2, ...)$ of real numbers that are *eventually zero*, i.e. for each $\underline{a}$ there is an N so that $a_n = 0$ for all $n \geq N$. Let d be the "usual" Euclidean distance:

$$d(\underline{a}, \underline{b}) = \sqrt{\sum_{n=1}^{\infty}(a_n - b_n)^2}$$

(a) Show that d is well-defined and $\mathbb{R}^\infty$ is a metric space.

(b) Find a closed and bounded subset of $\mathbb{R}^\infty$ that is **not** compact.

(c) Find a Cauchy sequence in $\mathbb{R}^\infty$ that does not converge.

4. (a) Prove directly that:

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{2n} = \sum_{n=0}^{\infty}\frac{2^n}{n!}$$

(b) Prove that this real number is e^2.

(c) Prove that e^2 is irrational.

5. Let X be a metric space and $f : X \to \mathbb{R}$ be a continuous function.

(a) If $E \subset X$ is compact, show that there exist $a \leq b \in \mathbb{R}$ such that:

(i) $f(p) = a$ and $f(q) = b$ for **some** $p, q \in E$ and
(ii) $a \leq f(x) \leq b$ for **all** $x \in E$

(b) If $F \subset X$ is connected and $p, q \in F$ satsify $f(p) = c < f(q) = d$, show that for every $e \in (c, d)$, there is an $x \in F$ such that $f(x) = e$.

6. (a) From the definition of the derivative (5.1) prove that

$$f : [a, b] \to \mathbb{R}$$

is differentiable at $x \in (a, b)$ with derivative $f'(x)$ if and only if the function $u(t)$ defined by: $f(t) = f(x) + (t - x)f'(x) + u(t)$ satisfies:

$$\lim_{t \to x} \frac{u(t)}{t - x} = 0$$

(b) State and prove the chain rule. (Yes, the proof is in the book. Try to rewrite it in your own words using (a))

(c) Use (b) to find the derivative of the inverse function $f^{-1}(y)$ at $y = f(x)$ in terms of $f'(x)$.

7. In the book, for the fixed closed interval $[0, 1]$, we let:

$$\mathcal{R} = \{\text{integrable (bounded) functions } f : [0, 1] \to \mathbb{R}\}$$

In this spirit, define:

$$\mathcal{C} = \{\text{continuous functions } f : [0, 1] \to \mathbb{R}\}$$

$$\mathcal{D} = \{\text{differentiable functions } f : [0, 1] \to \mathbb{R}\}$$

and $\mathcal{C}^1 = \{\text{differentiable functions with continuous derivatives}\}$.

(a) Prove each of the three inclusions:

$$\mathcal{C}^1 \subset \mathcal{D} \subset \mathcal{C} \subset \mathcal{R}$$

(b) For each of the inclusions in (a), find an explicit bounded function $f : [0, 1] \to \mathbb{R}$ that demonstrates that the inclusion is strict. Thus, for example, find a function that is differentiable, but whose derivative is not continuous to show that $\mathcal{C}^1 \neq \mathcal{D}$.

8. State the Mean Value Theorem for $f \in \mathcal{D}$ (for an arbitrary $[a, b]$) and explain in your own words how it is used in the proofs of:

(a) If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.

(b) l'Hôpital's rule

(c) Taylor's Theorem

(d) The Fundamental Theorem of Calculus

9. Suppose $f : [a, b] \to \mathbb{R}$ and $\alpha : [a, b] \to \mathbb{R}$ both have simple discontinuities at the same point $x \in (a, b)$ (and f is bounded and α is monotone increasing). Show that $f \notin \mathcal{R}(\alpha)$. Does the same necessarily hold true if the discontinuity of f is not simple?

10. Do Problem 10, (a)-(c) on Page 139 of Rudin.