1. Let $A$ be an operator on a finite-dimensional vector space $V$.
   (a) Prove that $A$ has a (nonzero, maybe complex) eigenvector.
   
   Let $B$ be another operator on $V$ that commutes with $A$, i.e.
   \[ AB = BA \]
   
   (b) Prove that $A$ and $B$ share a common (nonzero) eigenvector.
   
   Let $A_1, \ldots, A_n$ be commuting operators on $V$.
   
   (c) Prove that $A_1, \ldots, A_n$ share a common nonzero eigenvector.
   
   *Note.* In general, these eigenvectors will not have the same eigenvalues.

2. Suppose $A$ is an $n \times n$ matrix.
   (a) Define the Jordan normal form for $A$.
   (b) Sketch the proof that a suitable basis on $\mathbb{C}^n$ puts $A$ into Jordan normal form.

3. Consider the differential operator:
   \[ D_\alpha = \left( \frac{d}{dx} - \alpha \right) \]
   for some real number $\alpha$

   This, for example, satisfies:
   \[ D_\alpha(\sin(x)) = \frac{d}{dx} \sin(x) - \alpha \sin(x) = \cos(x) - \alpha \sin(x) \]
   
   (a) Find the one-dimensional kernel of the operator $D_\alpha$.
   (b) Find the $n$-dimensional kernel $V$ of the operator $(D_\alpha)^n$.
   (c) Show that $D_\alpha$ is an operator on the vector space $V$ from (b).
   (d) Find a cyclic vector in $V$ for the operator $D_\alpha$.

4. Find an invertible matrix $B$ so that $B^{-1}AB$ is in Jordan normal form, where:
   \[
   A = \begin{bmatrix}
   1 & 0 & 0 & 0 \\
   1 & 1 & 0 & 0 \\
   1 & 1 & 1 & 0 \\
   1 & 1 & 1 & 1 
   \end{bmatrix}
   \]

   What is the Jordan normal form?
5. For each of the following statements, either prove it or else give a counterexample.

(a) If all the eigenvalues of an $n \times n$ matrix are different from zero, the matrix is invertible.

(b) If an $n \times n$ matrix has zero as an eigenvalue, it is not invertible.

(c) The eigenvalues of a matrix with real entries are all real.

(d) Every symmetric $n \times n$ matrix has a real eigenbasis.

(e) Every real unitary $n \times n$ matrix has a real eigenbasis.

(f) Every $n \times n$ matrix has at most $n$ eigenvalues.

(g) If $P(t)$ is the characteristic polynomial of $A$, then $P(A) = 0$.  

6. Find a $5 \times 5$ matrix $A$ with all of the following properties:

$$ (A - 2I_5)^3(A - I_n)^2 = 0 $$

but

$$ (A - 2I_5)^2(A - I_n)^2 \neq 0 \text{ and } (A - 2I_5)^3(A - I_5) \neq 0 $$