Math 2200-002/Discrete Mathematics

Sequences and Series

Let S be a set.

Defintion. A *sequence* of elements of S is a function $f : \mathbb{N} \to S$. If $f(1) = a_1, f(2) = a_2, ..., f(n) = a_n$, then may write sequence as:

$$\{a_n\}_{n=1}^{\infty} \text{ or just } \{a_n\}$$

which is unfortunate because **a sequence is not a a set** since the elements of a sequence are *ordered* and may appear more than once.

Examples. (i) The sequence $\{a\}$ is the constant sequence $f(n) = a$.

(ii) The sequence:

$$f(1) = \text{Sunday}, f(2) = \text{Monday}, ..., f(7) = \text{Saturday},\ f(8) = \text{Sunday}, ...$$

is the days-of-the week sequence, which *cycles*, with $f(n+7) = f(n)$.

Some Sequences of Real Numbers

(A) A sequence of real numbers $\{a_n\}$ is *arithmetic* if:

$$a_n = d(n-1) + a \text{ for some real numbers } a \text{ and } d$$

That is, a_n is the sequence:

$$a, a+d, a+2d, \cdots$$

Some Familiar Arithmetic Sequences:

(i) The sequence of even numbers: $2, 4, 6, 8, ...$ $(a = 2, d = 2)$.

(ii) The sequence of odd numbers: $1, 3, 5, 7, ...$ $(a = 1, d = 2)$.

(iii) The sequence of negatives: $-1, -2, -3, ...$ $(a = -1, d = -1)$.

(B) A sequence of real numbers $\{a_n\}$ is *geometric* if:

$$a_n = ar^{n-1} \text{ for some real numbers } a, r$$

(and we usually require that $a \neq 0$).

Some Familiar Geometric Sequences:

(i) Compound Interest: $a = $ principal, $r = $ $1 +$ interest rate

(ii) Half lives: $a, \frac{1}{2}a, \frac{1}{4}a, \frac{1}{8}a, ...$

(iii) Doubling: $a, 2a, 4a, 8a, ...$

Definition. A *recurrence relation (RR)* for $\{a_n\}$ is a single! function:

$$a_n = g(a_{n-1}, ..., a_{n-k})$$

expresssing a_n as a function of a the previous k terms of the sequence.

Examples.

(i) An RR for the arithmetic sequence $a_n = a + (n-1)d$ is:

$$g(a_{n-1}) = a_{n-1} + d; \text{ i.e. } g(x) = x + d$$

(ii) An RR for the geometric sequence $a_n = ar^{n-1}$ is:

$$g(a_{n-1}) = ra_{n-1}; \text{ i.e. } g(x) = rx$$

(iii) An RR for the *Fibonacci sequence* $1, 1, 2, 3, 5, 8, 13, \ldots$ is:

$$g(a_{n-1}, a_{n-2}) = a_{n-1} + a_{n-2}; \text{ i.e. } g(x, y) = x + y$$

(this RR has $k = 2$, since it reaches back two terms).

Inductive Observation. If $g(a_{n-1}, .., a_{n-k})$ is an RR for the sequence $\{a_n\}$ that reaches back k terms, then the sequence $a_n = f(n)$ can be reconstructed from the function g and the first k terms of the sequence.

Example. (i) Arithmetic and geometric sequences are determined by their RRs and the first term of the sequence.

(ii) The Fibonacci sequence requires the first **two terms**

$$a_1 = 1, \ a_2 = 1$$

and the RR $g(a_{n-1}, a_{n-2}) = a_{n-1} + a_{n-2}$. A different two terms, e.g.

$$b_1 = 1, \ b_2 = 3$$

determine a different sequence with the same RR. In this case:

$$1, 3, 4, 7, 11, 18, \ldots$$

has a name. It is called the *Lucas* sequence.

An interesting (and sometimes hard) question is the following:

Question. How can we recover the function $f(n)$ for a sequence from the the RR function $g(a_{n-1}, ..., g_{n-k})$ and the initial k terms $a_1, .., a_k$?

Example. The RR $g(a_{n-1}) = a_{n-1} + d$ and $a_1 = a$ give:

$$f(n) = a + (n-1)d \text{ for arithmetic sequences}$$

Proposition. Let ϕ and ψ be the two roots of the quadratic relation:

$$x^2 = x + 1$$

That is, $\phi = (1 + \sqrt{5})/2$ (the golden mean) and $\psi = (1 - \sqrt{5})/2$. Then:

$$a_n = \frac{1}{\sqrt{5}}(\phi^n - \psi^n) \text{ for the Fibonacci sequence}$$

and $b_n = \phi^n + \psi^n$ for the Lucas sequence.

This remarkable fact gives a rapid convergence:

$$a_n \sim \phi^n/\sqrt{5} \text{ and } b_n \sim \phi^n$$

of the terms of the sequences which you should check on your calculator!

Proof. (Step 1.) The two geometric sequences:

$$\{\phi^n\}_{n-1}^{\infty} \text{ and } \{\psi^n\}_{n-1}^{\infty}$$

both satisfy the Fibonacci (and Lucas) RR:

$$a_n = a_{n-1} + a_{n-2}$$

since $x^2 = x + 1$ implies that $x^n = x^{n-1} + x^{n-2}$ (multiplying by x^{n-2}) in both the cases $x = \phi$ and $x = \psi$.

(Step 2.) Any **linear combination** of the two geometric sequences:

$$s\phi^n + t\psi^n$$

also satisfies the Fibonacci RR. Thus the proposition follows once we determine that the two sequences given by $s = 1/\sqrt{5}, t = -1/\sqrt{5}$ and $s = 1, t = 1$ match the first two terms of the Fibonacci and Lucas sequences, respectively. Using $\phi^2 = \phi + 1$ and $\psi^2 = \psi + 1$, we have:

$$\phi = (1 + \sqrt{5})/2, \psi = (1 - \sqrt{5})/2, \phi^2 = (3 + \sqrt{5})/2, \psi^2 = (3 - \sqrt{5})/2$$

so

$$\phi + \psi = 1 \text{ and } \phi^2 + \psi^2 = 3 \text{ matches the Lucas sequence!}$$

and

$$\phi - \psi = \sqrt{5} \text{ and } \phi^2 - \psi^2 = \sqrt{5}$$

matches $\sqrt{5}$ times Fibonacci. This proves the Proposition. □

Growth Rate is an important property of sequences:

(P) $\{a_n\}$ has *polynomial growth* if:

$$(\exists d \in \mathbb{N})(\exists C > 0)(\forall n >> 0)(|a_n| < Cn^d)$$

and the *minimal* value of d making this true is the degree of the growth.

Remark. $(\forall n >> 0)P(n)$ is mathematical shorthand for:

$$(\exists N \in \mathbb{N})(n > N \to P(n))$$

Linear Growth is polynomial growth of degree 1.

Quadratic growth is polynomial growth of degree 2.

Example. Arithmetic sequences have linear growth, but so do more sporadic sequences, like:

$$a_n = n + (-1)^n \text{ or } b_n = (-1)^n n$$

(E) $\{a_n\}$ has *exponential growth* if:

$$(\exists r > 0)(\exists C_1, C_2 > 0)(\forall n >> 0)(C_1 r^n < |a_n| < C_2 r^n)$$

and the single r making this true is the *rate* of the growth.

Examples. (a) The sequences $\{ar^n\}$ are exponential with rate r.

(b) Both the Fibonacci and Lucas sequences have exponential growth with growth rate ϕ.

Recall. From Calculus, we know that if $r > 1$ and $d > 0$, then:

$$\lim_{n\to\infty} \frac{r^n}{n^d} = \infty$$

(use l'Hopital's rule) so each exponential growth of rate $r > 1$ beats all polynomial growth rates.

When $r < 1$, the "growth" is called *exponential decay.*

Pseudo RRs. A pseudo-RR consists of recurrence relation functions:

$$a_n = g_n(a_{n-1}, ..., a_{n-k})$$

with some (simple) dependence on n. These PRRs plus the first k terms of the sequence still determine the sequence.

Examples. (a) The sequence of perfect squares $1, 4, 9, ...$ has:

$$a_n = a_{n-1} + 2n - 1$$

since $4 = 1 + 3$, $9 = 4 + 5$, etc. (see summations below).

(b) The **factorial** sequence $a_n = n!$ is defined by:

$$a_1 = 1 \text{ and } a_n = n \cdot a_{n-1}$$

The growth of this simple sequence beats all exponential growth rates!

Definition. Given a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers, let

$$\{s_n\}_{n=1}^{\infty}; \quad s_n = a_1 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

be the *sequence of partial sums.*

Examples. (i) If $a_n = 1$ is constant, then $s_n = n$ is arithmetic.

(ii) If $a_n = n$, then:

$$s_n = 1 + 2 + \cdots + n = \frac{(n+1)n}{2}$$

Proof. The sequence s_n has the PRR:

$$s_n = s_{n-1} + n \text{ and } s_1 = 1$$

But the sequence $b_n = (n+1)n/2$ also has $b_1 = 1$ and PRR:

$$b_n - b_{n-1} = \frac{(n+1)n}{2} - \frac{n(n-1)}{2} = n$$

so they are the same sequence! □

Corollary. The growth of s_n is *quadratic* if $\{a_n\}$ is arithmetic.

Let $a_n = d(n-1) + a$. Then by examples (i) and (ii), we have:

$$s_n = d \cdot \frac{n(n-1)}{2} + na = \left(\frac{d}{2}\right) n^2 + \left(\frac{2a-d}{2}\right) n$$

which is a quadratic polynomial in n.

Example. The sequence of perfect squares n^2 is the sequence of partial sums for the arithmetic sequence with: $d = 2$ and $2a - d = 0$ (so $a = 1$). That is, the arithmetic sequence is $a_n = 2(n-1) + 1 = 2n - 1$ and:

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

Fifth Homework Assignment. §2.3. #24,28,38,40. §2.4 #8,10,12,32,36,46.