The Chain Rule
Mathematics 1250-1 (Fall 2003)

If f and g are two functions that have derivatives, and if h is their **composition**:

$$h(x) = (f \circ g)(x) = f(g(x))$$

then the chain rule gives the derivative of h in terms of the derivatives of f and g. The rule is:

(Newton Notation) Let $u = g(x)$. Then:

$$h'(x) = f'(u) \cdot g'(x)$$

(Leibniz Notation) Let $y = f(u)$. Then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Thus when using the chain rule, you should first identify $g(x)$ and $f(u)$, then take the derivatives, multiply, and substitute $u = g(x)$ at the very end!

Example: Differentiate:

$$h(x) = \sqrt{x^3 + 1}$$

Identify the functions:

$$u = g(x) = x^3 + 1, \quad f(u) = \sqrt{u}$$

Differentiate:

$$g'(x) = 3x^2, \quad f'(u) = \frac{1}{2\sqrt{u}}$$

Multiply and substitute:

$$h'(x) = f'(u) \cdot g'(x) = \frac{1}{2\sqrt{u}} \cdot 3x^2 = \frac{1}{2\sqrt{x^3+1}} \cdot 3x^2 = \frac{3}{2} \cdot \frac{x^2}{\sqrt{x^3+1}}$$

Another Example: Differentiate:

$$h(x) = x^{-n} = \frac{1}{x^n}$$

Identify the functions:

$$u = g(x) = x^n, \quad f(u) = \frac{1}{u}$$

Differentiate:

$$g'(x) = nx^{n-1}, \quad f'(u) = -\frac{1}{u^2}$$

Multiply and substitute (and simplify):

$$h'(x) = f'(u) \cdot g'(x) = -\frac{1}{u^2} \cdot nx^{n-1} = -\frac{1}{x^{2n}} \cdot nx^{n-1} = -nx^{n-1-2n} = -nx^{-n-1}$$

A Consequence of the Chain Rule: $g(x)$ and $f(u)$ are **inverse functions** if:

$$f(g(x)) = x \text{ and } g(f(u)) = u$$

and in that case we will say that $f(u) = g^{-1}(u)$.

Now take the first equality and differentiate both sides:

$$f'(u) \cdot g'(x) = 1 \text{ (the derivative of } x \text{ is 1)}$$

This gives an important relation between inverse functions:

$$g'(x) = \frac{1}{f'(u)} = \frac{1}{(g^{-1})'(u)}$$

Example: $g(x) = x^{\frac{1}{n}}$ is the inverse function of $f(u) = u^n$. So:

$$g'(x) = \frac{1}{f'(u)} = \frac{1}{nu^{n-1}}$$

Now we substitute $u = x^{\frac{1}{n}}$ and simplify:

$$g'(x) = \frac{1}{n(x^{\frac{1}{n}})^{n-1}} = \frac{1}{nx^{1-\frac{1}{n}}} = \frac{1}{n}x^{\frac{1}{n}-1}$$

Notice: We have found the derivatives:

$$\frac{d}{dx}(x^n) = nx^{n-1}, \quad \frac{d}{dx}(x^{-n}) = -nx^{-n-1} \text{ and } \frac{d}{dx}(x^{\frac{1}{n}}) = \frac{1}{n}x^{\frac{1}{n}-1}$$

Claim: For all the fractional powers of x:

$$\frac{d}{dx}(x^{\frac{m}{n}}) = \frac{m}{n}x^{\frac{m}{n}-1}$$

Proof: Use the chain rule. If $h(x) = x^{\frac{m}{n}}$:

Identify the functions:

$$h(x) = f(g(x)) \text{ for } u = g(x) = x^{\frac{1}{n}} \text{ and } f(u) = u^m$$

Differentiate:

$$g'(x) = \frac{1}{n}x^{\frac{1}{n}-1} \text{ and } f'(u) = mu^{m-1}$$

Multiply and substitute (and simplify):

$$h'(x) = f'(u) \cdot g'(x) = (mu^{m-1})(\frac{1}{n}x^{\frac{1}{n}-1}) = \frac{m}{n} \cdot x^{\frac{m-1}{n}+\frac{1}{n}-1} =$$

$$\frac{m}{n}x^{\frac{m}{n}-\frac{1}{n}+\frac{1}{n}-1} = \frac{m}{n}x^{\frac{m}{n}-1}$$

Voilà!