Pseudo-random numbers: a line (of code) at a time (mostly)

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What are random numbers good fo

- Decision making (e.g., coin flip).
- Generation of numerical test data.
- Generation of unique cryptographic key
- Search and optimization via random wa
- Selection: `quicksort` (C. A. R. Hoare, *Quicksort*, Comm. ACM. **4**(7), 321, Ju widely-used divide-and-conquer *algorith* problem to (on average) $\mathcal{O}(N \lg(N))$. (Clairaut (1754), Lagrange (1759), Gau [Latin], Runge (1903), Danielson and L (1942), Cooley and Tukey (1965)).

Historical note: al-Khwarizmi

Abu 'Abd Allah Muhammad ibn Musa al-Khwarizmi (ca. 780–850) is the father of *algorithm* and of *algebra*, from his book *Hisab Al-Jabr wal Mugabalah (Book of Calculations, Restoration and Reduction)*. He is celebrated in a 1200-year anniversary Soviet Union stamp:

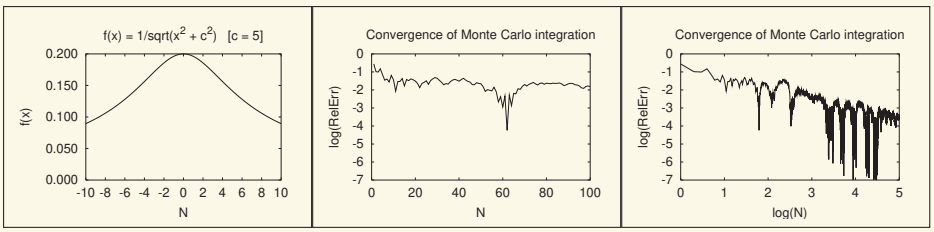
What are random numbers good fo

- Simulation.
- Sampling: unbiased selection of random computations (opinion polls, experimen Monte Carlo integration, ...). The latt random in (a, b)):

$$\int_a^b f(x)\,dx \approx \left(\frac{(b-a)}{N} \sum_{k=1}^{N}$$

Monte Carlo integration

Here is an example of a simple, smooth, and exactly integrable function, and the relative error of its Monte Carlo integration:



When is a sequence of numbers ra

- Computer numbers are rational, with li
  Irrational and transcendental numbers a
- Truly random integers would have occa
  pseudo-random number generators proc
  the *period*, of distinct integers: these c
- It isn't enough to conform to an expect
  values appear in must be haphazard.
- Mathematical characterization of rando
- The best that we can usually do is *com*
  closeness to particular expected distribu

Distributions of pseudo-random numbers

- Uniform (most common).
- Exponential.
- Normal (bell-shaped curve).
- Logarithmic: if $\mathrm{ran}()$ is uniformly-distributed in (a, b), define $\mathrm{randl}(x) = \exp(x\,\mathrm{ran}())$. Then $a\,\mathrm{randl}(\ln(b/a))$ is logarithmically distributed in (a, b). [Important use: sampling in floating-point number intervals.]

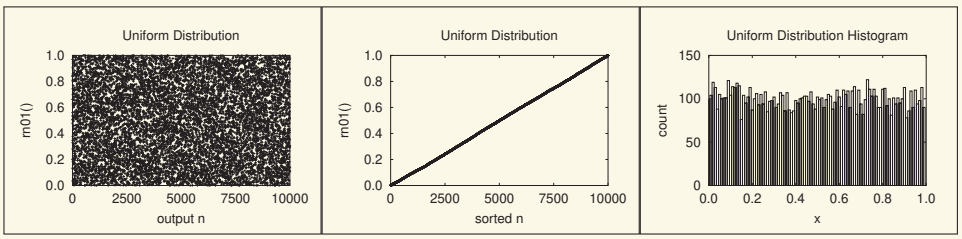
Distributions of pseudo-random nu

Sample logarithmic distribution:

```
% hoc
a = 1
b = 1000000
for (k = 1; k <= 10; ++k) printf "%
    664.28612484
 199327.86997895
 562773.43156449
  91652.89169494
     34.18748767
    472.74816777
     12.34092778
      2.03900107
  44426.83813202
     28.79498121
```

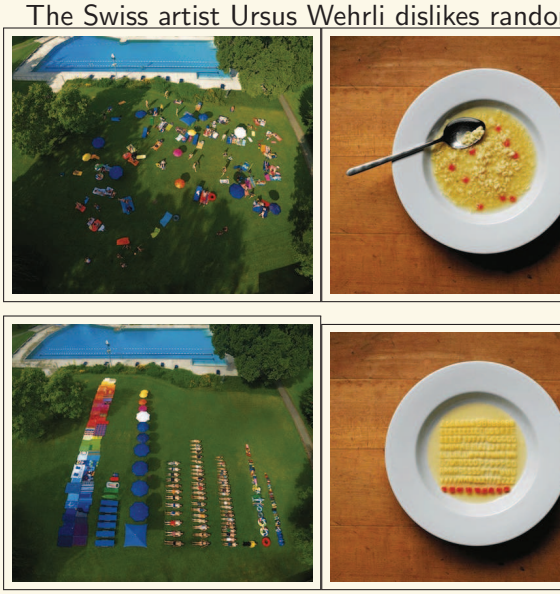
Uniform distribution

Here are three ways to visualize a pseudo-random number distribution, using the Dyadkin-Hamilton generator function `rn01()`, which produces results uniformly distributed on $(0, 1]$:



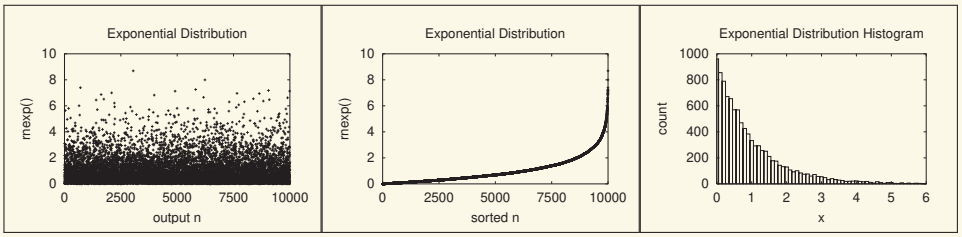
Disorder and order

The Swiss artist Ursus Wehrli dislikes rando



Exponential distribution

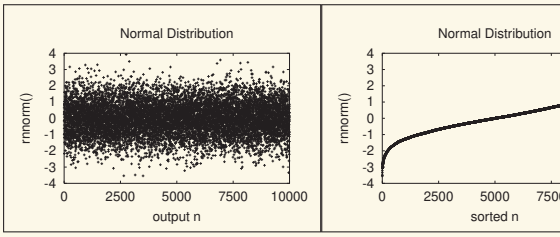
Here are visualizations of computations with the Dyadkin-Hamilton generator `rnexp()`, which produces results exponentially distributed on $[0, \infty)$:



Even though the theoretical range is $[0, \infty)$, the results are practically always modest: the probability of a result as big as 50 is smaller than 2×10^{-22}. At one result per microsecond, it could take 164 million years of computing to encounter such a value!

Normal distribution

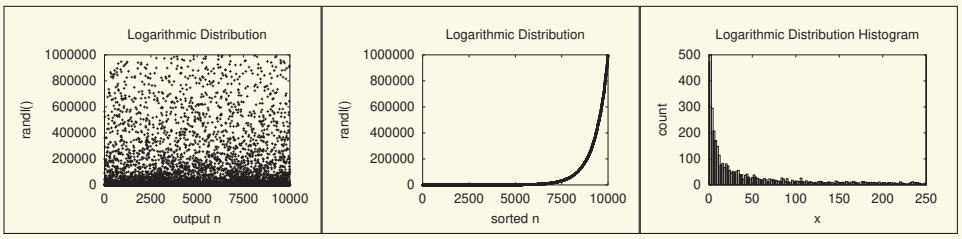
Here are visualizations of computations with generator `rnnorm()`, which produces results $(-\infty, +\infty)$:



Results are never very large: a result as big smaller than 5×10^{-23}. At one result per m 757 million years of computing to encounter

Logarithmic distribution

Here are visualizations of computations with the `hoc` generator `randl(ln(1000000))`, which produces results normally distributed on $(1, 1000000)$:



The graphs are similar to those for the exponential distribution, but here, the result range is controlled by the argument of `randl()`.

Goodness of fit: the χ^2 measure

Given a set of *n independent observations* w
expected values E_k, then $\sum_{k=1}^{n} |(E_k - M_k)|$
fit. So is $\sum_{k=1}^{n} (E_k - M_k)^2$. Statisticians use
in 1900 by one of the founders of modern st
mathematician Karl Pearson (1857–1936):

(1880)

χ^2 measure

Equivalently, if we have
with probability p_k, and
the number Y_k in cate

χ^2 measure

Goodness of fit: the χ^2 measure . . .

The theoretical χ^2 distribution depends on the number of degrees of freedom, and table entries look like this (highlighted entries are referred to later):

D.o.f.	$p = 1\%$	$p = 5\%$	$p = 25\%$	$p = 50\%$	$p = 75\%$	$p = 95\%$	$p = 99\%$
$\nu = 1$	0.00016	0.00393	0.1015	**0.4549**	1.323	3.841	**6.635**
$\nu = 5$	0.5543	1.1455	2.675	4.351	6.626	11.07	15.09
$\nu = 10$	2.558	3.940	6.737	9.342	12.55	18.31	**23.21**
$\nu = 50$	29.71	34.76	42.94	49.33	56.33	67.50	76.15

For example, this table says:

> **For $\nu = 10$, the probability that the χ^2 measure is no larger than 23.21 is 99%.**
> **In other words, χ^2 measures larger than 23.21 should occur only about 1% of the time.**

Goodness of fit: coin-toss experime

Coin toss has one degree of freedom, $\nu =$
then it must be tails.

```
% hoc
for (k = 1; k <= 10; ++k) print randint(0,1
0 1 1 1 0 0 0 0 1 0
```

This gave four 1s and six 0s:

$$\chi^2 \text{ measure} = \frac{(10 \times 0.5 - 4)}{10}$$
$$= 2/5$$
$$= 0.40$$

Goodness of fit: coin-toss experiments . . .

From the table, for $\nu = 1$, we expect a χ^2 measure no larger than **0.4549** half of the time, so our result is reasonable.
On the other hand, if we got nine 1s and one 0, then we have

$$\begin{aligned} \chi^2 \text{ measure} &= \frac{(10 \times 0.5 - 9)^2 + (10 \times 0.5 - 1)^2}{10 \times 0.5} \\ &= 32/5 \\ &= 6.4 \end{aligned}$$

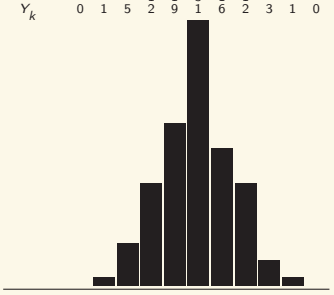
This is close to the tabulated value **6.635** at $p = 99\%$. That is, **we should only expect nine-of-a-kind about once in every 100 experiments.**
If we had all 1s or all 0s, the χ^2 measure is 10 (probability $p = 0.998$) [**twice in 1000 experiments**].
If we had equal numbers of 1s and 0s, then the χ^2 measure is 0, indicating an exact fit.

Goodness of fit: coin-toss experiments . . .

The measured frequencies of the sums are:

100 experiments

k	0	1	2	3	4	5	6	7	8	9	10
Y_k	0	1	5	12	19	31	16	12	3	1	0



Notice that nine-of-a-kind occurred **once** each for 0s and 1s, as predicted.

Goodness of fit: coin-toss experime

Let's try 100 similar experiments, counting t
experiment:

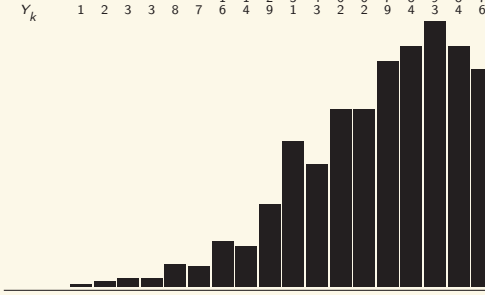
```
% hoc
for (n = 1; n <= 100; ++n) {
    sum = 0
    for (k = 1; k <= 10; ++k) \
        sum += randint(0,1)
    print sum, ""
}
4 4 7 3 5 5 5 2 5 6 6 6 3 6 6 7 4 5
3 6 6 9 5 3 4 5 4 4 4 5 4 5 5 4 6 3
4 7 2 6 5 3 6 5 6 7 6 2 5 3 5 5 5 7
8 4 2 7 7 3 3 5 4 7 3 6 2 4 5 1 4 5
5 6 5 5 4 8 7 7 5 5 4 5
```


Goodness of fit: coin-toss experime

A simple one-character change on the outer
experiment:

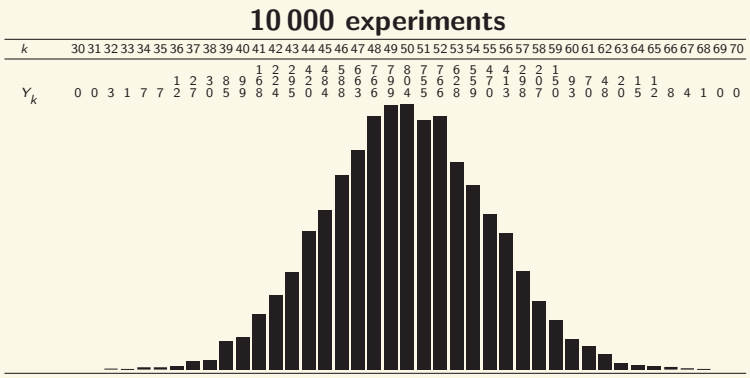
1000 experime

k	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52
Y_k	1	2	3	3	8	7	16	14	29	51	43	62	79	84	93	84	6	7



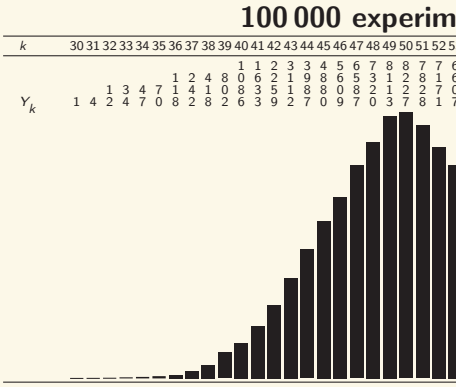
Goodness of fit: coin-toss experiments . . .

Another one-character change gives us this:



Goodness of fit: coin-toss experime

A final one-character change gives us this re



Are the digits of π random?

Here are χ^2 results for the digits of π from recent computational records ($\chi^2(\nu = 9, \mathrm{p} = 0.99) \approx 21.67$):

π			
Digits	**Base**	χ^2	$\mathrm{p}(\chi^2)$
6B	10	9.00	0.56
50B	10	5.60	0.22
200B	10	8.09	0.47
1T	10	14.97	0.91
1T	16	7.94	0.46

$1/\pi$			
Digits	**Base**	χ^2	$\mathrm{p}(\chi^2)$
6B	10	5.44	0.21
50B	10	7.04	0.37
200B	10	4.18	0.10

Whether the fractional digits of π, and most other transcendentals, are *normal* ($\approx$ equally likely to occur) is an outstanding unsolved problem in mathematics.

Are the first 1000 fractional digits

3.14159265358979323846264338327950288419
71693993751058209749445923078164062862089
98628034825342117067982148086513282306647
09384460955058223172535940812848111745028
41027019385211055596446229489549303819644
28810975665933446128475648233786783165271
20190914564856692346034861045432664821339
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20920962829254091715364367892590360011330
53054882046652138414695194151160943305727
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48074462379962749567351885752724891227938
18301194912983367336244065664308602139494
63952247370707831605693034207754177294088
10241809415526601747855432178434636914347
61328992738190405188026004613449470495378
23168636945405893509860187145808577263243
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54355752133475254241686997672523850836688
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74518463686251021967569751773671892589402
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Are the first 1000 fractional digits of π random?

In the first 1000 fractional digits of π:

- 83 digit pairs (81 expected)
- 77-digit sequence without a 4 (probability: $(9/10)^{77} \approx 0.0003$)
- six consecutive 9 digits (probability: 1/1,000,000)
- last five digits are a calendar year (probability: 1/100,000)

Conclusion: for a **finite** sequence of digits, the answer is **no!**
See Aaldert Compagner, *Definitions of randomness*, American Journal of Physics **59**(8) 700–705 (1991).
URL `http://m.ajp.aapt.org/resource/1/ajpias/v59/i8/p700_s1`

Is your name in π?

From `http://www.dr-mikes-maths.com/`
NELSON was not found, but I searched 31 4
BEEBE 4 times. The first occurrence was a
means is that

$$\pi = 3 + \ldots + \frac{B}{27^{278246052}} + \frac{E}{27^{278246053}} + \frac{E}{27^{278246054}} +$$

where $A = 1$, $B = 2$, $C = 3$, and so on.

The Central-Limit Theorem

The famous **Central-Limit Theorem** (de Moivre (1718), Laplace (1810), and Cauchy (1853)), says:

> **A suitably normalized sum of independent random variables is likely to be normally distributed, as the number of variables grows beyond all bounds. It is not necessary that the variables all have the same distribution function or even that they be wholly independent.**
>
> *— I. S. Sokolnikoff and R. M. Redheffer*
> *Mathematics of Physics and Modern Engineering, 2nd ed.*

The Central-Limit Theorem ...

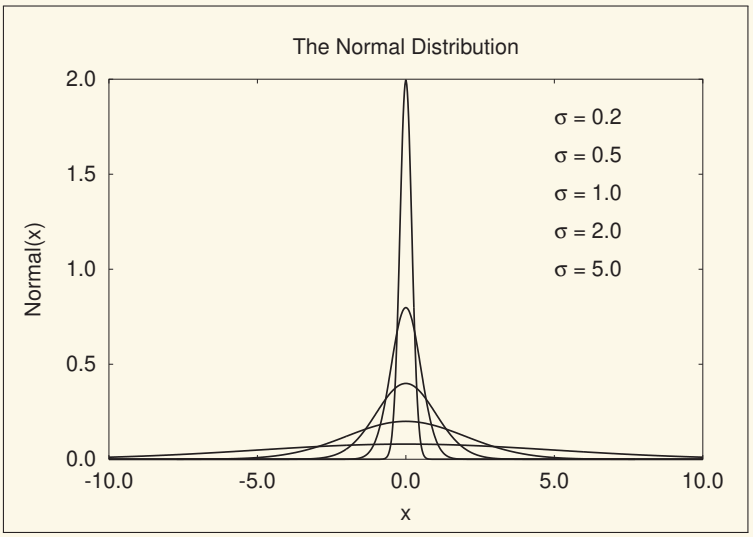
In mathematical terms, this is

$$P(n\mu + r_1\sqrt{n} \le X_1 + X_2 + \cdots +$$
$$\approx \frac{1}{\sigma\sqrt{2\pi}} \int_{r_1}^{r_2} \exp(-t^2$$

where the X_k are independent, identically di
random variables, μ is their **mean value**, σ
and σ^2 is their **variance**.

The Central-Limit Theorem ...

The integrand of this probability function looks like this:



The Central-Limit Theorem ...

The normal curve falls off very rapidly. We
$[-x, +x]$ with a simple midpoint quadrature

```
func f(x) {
    global sigma;
    return (1/(sigma*sqrt(2*PI)))*
}

func q(a,b) {
    n = 10240
    h = (b - a)/n
    area = 0
    for (k = 0; k < n; ++k) \
         area += h*f(a + (k + 0.5)*
    return area
}
```

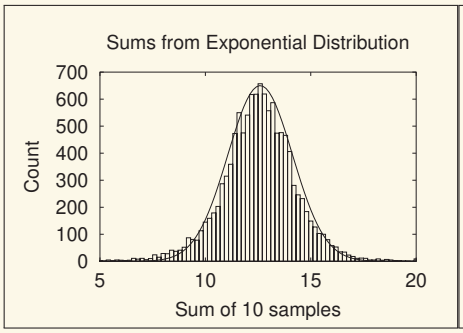

The Central-Limit Theorem ...

```
sigma = 3
for (k = 1; k < 8; ++k) \
    printf "%d  %.9f\n", k, q(-k*sigma,k*sigma)
1  0.682689493
2  0.954499737
3  0.997300204
4  0.999936658
5  0.999999427
6  0.999999998
7  1.000000000
```

In computer management, **99.999% (five 9's) availability** is **five minutes downtime per year**.
In manufacturing, Motorola's **6σ reliability** with 1.5σ drift is about **three defects per million** (from $q(-(6-1.5)*\sigma, +(6-1.5)*\sigma)/2$).

The Central-Limit Theorem ...

It is remarkable that the Central-Limit Theo
distributions. Here is a demonstration with
normal distributions:



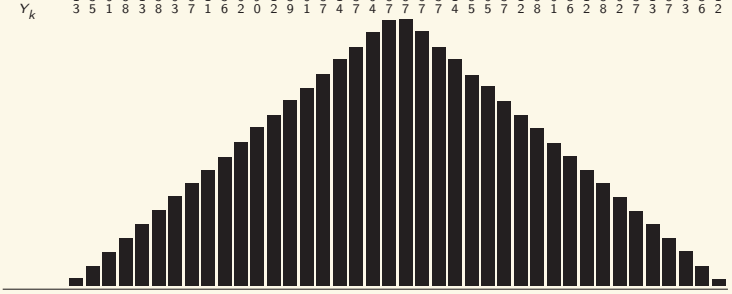
Superimposed on the histograms are rough
distribution curves $650\exp(-(x-12.6)^2/4$
$550\exp(-(x-13.1)^2/2.3)$.

The Central-Limit Theorem . . .

Not everything looks like a normal distribution. Here is a similar experiment, using *differences* of successive pseudo-random numbers, bucketizing them into 40 bins from the range $[-1.0, +1.0]$:

10 000 experiments (counts scaled by 1/100)

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
Y_k	13	35	61	81	113	138	163	187	211	236	260	290	312	329	361	381	414	437	474	487	477	467	474	414	385	357	337	312	281	261	236	212	182	163	137	117	87	63	36	12



This one is known from theory: it is a *triangular* distribution. A similar result is obtained if one takes pair sums instead of differences.

Digression: Poisson distribution

The *Poisson* distribution arises in time series event occurring in an arbitrary interval is pr interval, and independent of other events:

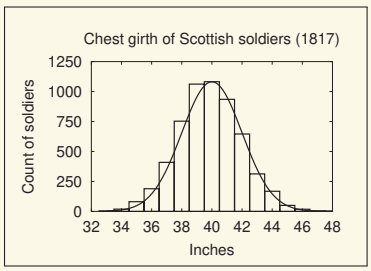
$$P(X = n) = \frac{\lambda^n}{n!}$$

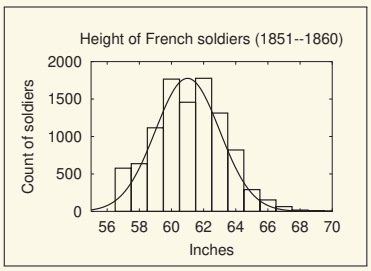
In 1898, Ladislaus von Bortkiewicz collected number of soldiers killed by horse kicks in 1 122 deaths, or an average of $122/200 = 0.$

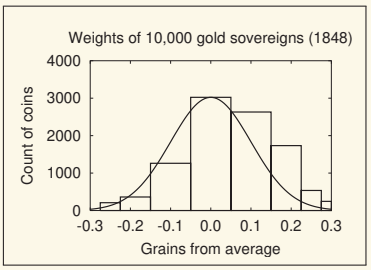
	$\lambda = 0.61$	
Deaths	Kicks (actual)	Kicks (Poisson)
0	109	108.7
1	65	66.3
2	22	20.2
3	3	4.1
4	1	0.6

The Central-Limit Theorem . . .

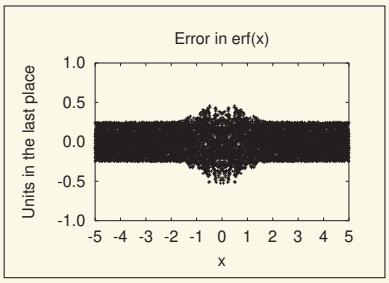
Measurements of physical phenomena often form normal distributions:

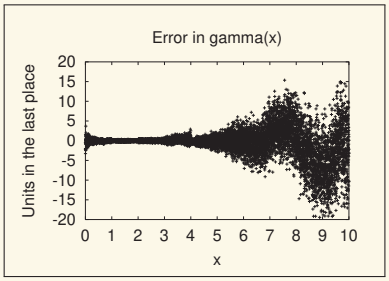




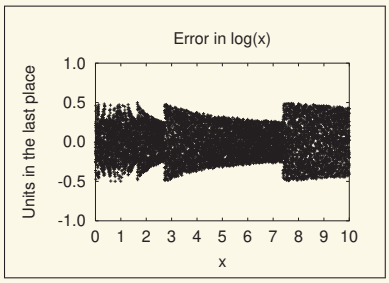


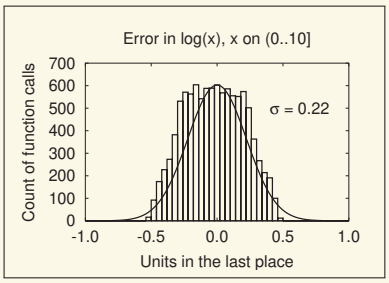
The Central-Limit Theorem . . .

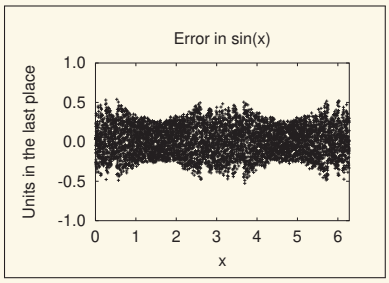


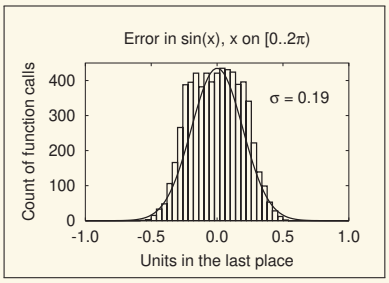


The Central-Limit Theorem . . .



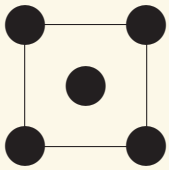






The Normal Curve and Carl-Friedri

The Normal Curve and the Quincunx

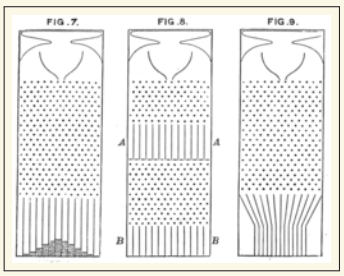


quincunx, *n.*

2. An arrangement or disposition of five objects so placed that four occupy the corners, and the fifth the centre, of a square or other rectangle; a set of five things arranged in this manner.
b. spec. as a basis of arrangement in planting trees, either in a single set of five or in combinations of this; a group of five trees so planted.

Oxford English Dictionary

The Normal Curve and the Quincu



For simulations and other material on the q
machine), see:

- `http://www.ms.uky.edu/~mai/java`
- `http://www.rand.org/statistics/`
- `http://www.stattucino.com/berri`
- `http://teacherlink.org/content/`
  `flash/quincunx/quincunx.html`
- `http://www.bun.kyoto-u.ac.jp/~s`

Remarks on random numbers

Any one who considers arithmetical methods of producing random numbers is, of course, in a state of sin.
— John von Neumann (1951)
[*The Art of Computer Programming, Vol. 2, Seminumerical Algorithms*, 3rd ed., p. 1]

He talks at random; sure, the man is mad.
— Queen Margaret
[William Shakespeare's *1 King Henry VI*, Act V, Scene 3 (1591)]

A random number generator chosen at random isn't very random.
— Donald E. Knuth (1997)
[*The Art of Computer Programming, Vol. 2, Seminumerical Algorithms*, 3rd ed., p. 384]

How do we generate pseudo-rando

- Linear-congruential generators (most co
  $r_{n+1} = (ar_n + c) \bmod m$, for integers a
  $0 \le a < m$, $0 \le c < m$, with starting v
- Fibonacci sequence (bad!):
  $r_{n+1} = (r_n + r_{n-1}) \bmod m$.
- Additive (better): $r_{n+1} = (r_{n-\alpha} + r_{n-\beta}$
- Multiplicative (bad):
  $r_{n+1} = (r_{n-\alpha} \times r_{n-\beta}) \bmod m$.
- Shift register:
  $r_{n+k} = \sum_{i=0}^{k-1} (a_i r_{n+i} \pmod 2)$ $(a_i$

How do we generate pseudo-random numbers? ...

Given an integer $r \in [A, B)$, $x = (r - A)/(B - A + 1)$ is on $[0, 1)$. However, interval reduction by $A + (r - A) \bmod s$ to get a distribution in (A, C), where $s = (C - A + 1)$, is possible only for certain values of s. Consider reduction of $[0, 4095]$ to $[0, m]$, with $m \in [1, 9]$: we get equal distribution of remainders only for $m = 2^q - 1$:

	m	counts of remainders k mod $(m+1)$, $k \in [0, m]$									
OK	1	2048	2048								
	2	**1366**	1365	1365							
OK	3	1024	1024	1024	1024						
	4	**820**	819	819	819	819					
	5	**683**	**683**	**683**	**683**	682	682				
	6	**586**	585	585	585	585	585	585			
OK	7	512	512	512	512	512	512	512	512		
	8	**456**	455	455	455	455	455	455	455	455	
	9	**410**	**410**	**410**	**410**	**410**	**410**	409	409	409	409

How do we generate pseudo-rando

Samples from other distributions can usually transformation. Here is the simplest generat assuming that `randu()` returns uniformly-di

```
func randpmnd() \
{ ## Polar method for random deviat
  ## Algorithm P, p. 122, from Dona
  ## The Art of Computer Programmin
  while (1) \
  {
    v1 = 2*randu() - 1  # v1 on [-1
    v2 = 2*randu() - 1  # v2 on [-1
    s = v1*v1 + v2*v2   # s on [0,2
    if (s < 1) break    # exit loop
  }
  return (v1 * sqrt(-2*ln(s)/s))
}
```

Period of a sequence

All pseudo-random number generators eventually reproduce the starting sequence; the *period* is the number of values generated before this happens.
Widely-used historical generators have periods of a few tens of thousands to a few billion, but good generators are now known with very large periods:

- $> 10^{14}$ POSIX `drand48()` LCG (2^{48}) (1982),
- $> 10^{57}$ Marsaglia short and fast `xorshift()` (2^{192}) (2003),
- $> 10^{18}$ Numerical Recipes `ran2()` (1992),
- $> 10^{38}$ NIST Advanced Encryption Standard (AES) (2^{128}) (2003),
- $> 10^{449}$ Matlab's `rand()` ($\approx 2^{1492}$ Columbus generator),
- $> 10^{2894}$ Marsaglia's Monster-KISS (2000),
- $> 10^{6001}$ Matsumoto and Nishimura's Mersenne Twister (1998),
- $> 10^{14100}$ Deng and Xu (2003),
- $> 10^{16736}$ Berdnikov, Trutia, & Compagner `MathLink` (1996).

Reproducible sequences

In computational applications with pseudo-r
to be able to reproduce a previous calculatio
required that can be set to a given **initial s**

```
% hoc
for (k = 0; k < 3; ++k) \
{
    setrand(12345)
    for (n = 0; n < 10; ++n) print int
    println ""
}
88185 5927 13313 23165 64063 90785 240
88185 5927 13313 23165 64063 90785 240
88185 5927 13313 23165 64063 90785 240
```


Reproducible sequences . . .

If the seed is not reset, different sequences are obtained for each test run. Here is the same code as before, with the `setrand()` call disabled:

```
for (k = 0; k < 3; ++k) \
{
    ## setrand(12345)
    for (n = 0; n < 10; ++n) print int(rand()*100000),""
    println ""
}
36751 37971 98416 59977 49189 85225 43973 93578 61366 54404
70725 83952 53720 77094 2835 5058 39102 73613 5408 190
83957 30833 75531 85236 26699 79005 65317 90466 43540 14295
```

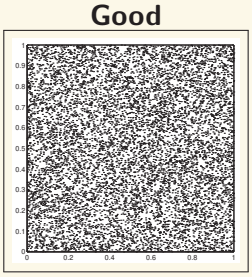
In practice, **software must have its own source-code implementation of the generators**: vendor-provided ones do *not* suffice.

The correlation problem

Random numbers

Linear-congruential generators are known to
numbers: if these are used as coordinates in
instead of uniform grey:

Good



The number of points plotted is the san

The correlation problem . . .

The good generator is Matlab's `rand()`. Here is the bad generator:

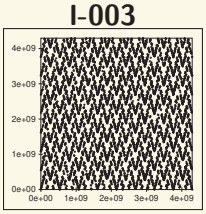
```
% hoc
func badran() {
    global A, C, M, r;
    r = int(A*r + C) % M;
    return r }
M = 2^15 - 1; A = 2^7 - 1 ; C = 2^5 - 1
r = 0 ; r0 = r ; s = -1 ; period = 0

while (s != r0) {period++; s = badran(); print s, "" }
    31 3968 12462 9889 10788 26660 ... 22258 8835 7998 0

# Show the sequence period
println period
    175

# Show that the sequence repeats
for (k = 1; k <= 5; ++k) print badran(),""
    31 3968 12462 9889 10788
```


The correlation problem . . .

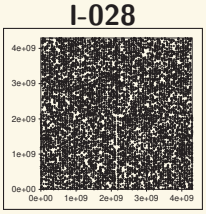
Marsaglia's (2003) family of xor-shift genera

```
y ^= y << a; y ^= y >> b; y ^= y <<
```

I-003



I-028



Generating random integers

When the endpoints of a floating-point uniform pseudo-random number generator are uncertain, generate random integers in [`low`,`high`] like this:

```
func irand(low, high) \
{
    # Ensure integer endpoints
    low = int(low)
    high = int(high)

    # Sanity check on argument order
    if (low >= high) return (low)

    # Find a value in the required range
    n = low - 1
    while ((n < low) || (high < n)) \
        n = low + int(rand() * (high + 1 - low))

    return (n)
}


for (k = 1; k <= 20; ++k) print irand(-9,9), ""
-9 -2 -2 -7 7 9 -3 0 4 8 -3 -9 4 7 -7 8 -3 -4 8 -4

for (k = 1; k <= 20; ++k) print irand(0, 10^6), ""
986598 580968 627992 379949 700143 734615 361237
322631 116247 369376 509615 734421 321400 876989
940425 139472 255449 394759 113286 95688
```


Generating random integers in orde

```
% hoc
func bigrand() { return int(2^31 * rand()) }

# select(m,n): select m pseudo-random integers from (0,n) in or
proc select(m,n) \
{
    mleft = m
    remaining = n
    for (i = 0; i < n; ++i) \
    {
        if (int(bigrand() % remaining) < mleft) \
        {
            print i, ""
            mleft--
        }
        remaining--
    }
    println ""
}
```

See Chapter 12 of Jon Bentley, *Programmin* Addison-Wesley (2000), ISBN 0-201-65788- 359–364, September 1980].

Generating random integers in order . . .

Here is how the `select()` function works:

```
select(3,10)
5 6 7

select(3,10)
0 7 8

select(3,10)
2 5 6

select(3,10)
1 5 7

select(10,100000)
7355 20672 23457 29273 33145 37562 72316 84442 88329 97929

select(10,100000)
401 8336 41917 43487 44793 56923 61443 90474 92112 92799
```


Testing pseudo-random number ge

Most tests are based on computing a χ^2 me
theoretical values.
**If one gets values p $<$ 1% or p $>$ 99% f
generator is suspect.**
Marsaglia Diehard Battery test suite (1985)
Marsaglia/Tsang `tuftest` suite (2002): 3 t
All produce p values that can be checked fo
These tests all expect *uniformly-distributed*

Testing nonuniform pseudo-random number generators

How do you test a generator that produces pseudo-random numbers in some other distribution? You have to figure out a way to use those values to produce an expected uniform distribution that can be fed into the standard test programs.
For example, take the negative log of exponentially-distributed values, since $-\log(\exp(-\text{random})) = \text{random}$.
For normal distributions, consider successive pairs (x, y) as a 2-dimensional vector, and express in polar form (r, θ): θ is then uniformly distributed in $[0, 2\pi)$, and $\theta/(2\pi)$ is in $[0, 1)$.

The Marsaglia/Tsang `tuftest` tes

Just three tests instead of the fifteen of the

- b'day test (generalization of Birthday P
- Euclid's (ca. 330–225BC) gcd test.
- Gorilla test (generalization of monkey's characters).

Digression: The Birthday Paradox

The *birthday paradox* arises from the question **How many people do you need in a room before the probability is at least half that two of them share a birthday?**
The answer is just 23, not $365/2 = 182.5$.
The probability that *none* of n people are born on the same day is

$$\begin{aligned} P(1) &= 1 \\ P(n) &= P(n-1) \times (365 - (n-1))/365 \end{aligned}$$

The n-th person has a choice of $365 - (n-1)$ days to not share a birthday with any of the previous ones. Thus, $(365 - (n-1))/365$ is the probability that the n-th person is not born on the same day as any of the previous ones, assuming that they are born on different days.

Digression: The Birthday Paradox

Here are the probabilities that n people *shar*

```
% hoc128
PREC = 3
p = 1
for (n = 1;n <= 365;++n) \
    {p *= (365-(n-1))/365; println n,1-p}
1 0
2 0.00274
3 0.00820
4 0.0164
...
22 0.476
23 0.507
24 0.538
...
100 0.999999693
...
```

$P(365) \approx 1.45 \times 10^{-157}$ [cf. 10^{80} particles i

Digression: Euclid's algorithm (ca. 300BC)

This is the oldest surviving nontrivial algorithm in mathematics.

```
func gcd(x,y) \
{ ## greatest common denominator of integer x, y
  r = abs(x) % abs(y)
  if (r == 0) return abs(y) else return gcd(y, r)
}

func lcm(x,y) \
{ ## least common multiple of integer x,y
  x = int(x)
  y = int(y)
  if ((x == 0) || (y == 0)) return (0)
  return ((x * y)/gcd(x,y))
}
```


Digression: Euclid's algorithm ...

Complete rigorous analysis of Euclid's algori
1970–1990!
The average number of steps is

$$\begin{aligned} A(\gcd(x,y)) &\approx ((12 \ln \\ &\approx 1.9405 \end{aligned}$$

and the maximum number is

$$\begin{aligned} M(\gcd(x,y)) &= \lfloor \log_\phi ((\\ &\approx 4.785 \log \end{aligned}$$

where $\phi = (1+\sqrt{5})/2 \approx 1.6180$ is the gol

Hardware generators

- Cryptographic accelerator boards (IBM, Sun/Oracle, ...)
- Quantis PCI card (2004) (`http://www.randomnumbers.info/`)
  863 328 405 985 310 188 300 795 5 886
  84 210 411 664 264 438 221 561 756 152
  617 652 112 316 551 102 682 2 851 425
- New Intel hardware: `RdRand` for 16-, 32-, and 64-bit random values [IEEE Spectrum September 2011]