New directions in floating-point arithmetic

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26 September 2007
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- IEEE 754 design first implemented in Intel 8087 coprocessor (1980)
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Floating-point arithmetic can make error analysis difficult, with behavior like this in some older designs:

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- poor rounding practices increase cumulative rounding error
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- ±0, ±∞, signaling and quiet NaN
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- NaN from 0/0, $\infty - \infty$, $f(\text{NaN})$, $x \text{ op } \text{NaN}$, ...
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- approximate precisions in decimal digits: 7, 15, 19, 34, 70
- approximate ranges (powers of 10): $[-45, 38]$, $[-324, 308]$, $[-4951, 4932]$, $[4966, 4932]$, $[-315\ 723, 315\ 652]$
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- some platforms have nonconforming rounding behavior
Why the base matters

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- for a fixed storage size, granularity (the spacing between successive representable numbers) increases as the base increases
- in the absence of underflow and overflow, multiplication by a power of the base is an exact operation, and this feature is essential for many computations, in particular, for accurate elementary and special functions
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- exact in one base may be inexact in others (e.g., decimal 0.9 is hexadecimal 0x1.ccccccccccccccccccccccccccccc...p-1)
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- few (if any) languages guarantee accurate base conversion
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IBM Densely-Packed Decimal (DPD) and Intel Binary-Integer Decimal (BID) in 32-bit, 64-bit, 128-bit, and 256-bit formats provide $3n + 1$ digits: 7, 16, 34, and 70
Decimal floating-point arithmetic

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<td>0 1</td>
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Nelson H. F. Beebe (University of Utah)  New directions in floating-point arithmetic  26 September 2007
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- Scripting languages: gawk, hoc, lua, mawk, nawk
Virtual platforms

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