# RESEARCH STATEMENT

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My research has been on questions in algebraic geometry that are at least indirectly motivated by physics. Algebraic geometry studies algebraic varieties, i.e. spaces that can (at least locally) be described as the zero sets of polynomial equations. In the last 20 years, it has had a close interaction with string theory, where algebraic varieties appear as target spaces in so–called "sigma models".

A guiding principle of my recent research has been the study of "wall-crossing" in various settings; this term describes the following rather general phenomenon:

In order to successfully study objects (curves, varieties, vector bundles, coherent sheaves) in algebraic geometry, one typically also has to study families of such objects. This can often be reduced to studying the universal family of such objects parametrized by a universal space, called moduli space in algebraic geometry. However, in order to get a well-behaved moduli space (for example, to get a compact moduli space with a Hausdorff topology), one only allows a subset of objects, satisfying a certain stability condition $\alpha$.

Often, there is no natural choice of a stability condition. However, only recently have algebraic geometers begun to systematically study how the moduli space changes when we vary the stability condition $\alpha$. Typically, the moduli space remains unchanged while $\alpha$ varies in an open chamber of the set of possible stability conditions, but jumps once $\alpha$ crosses a wall.

I have studied this phenomenon in the setting of objects in the derived category of coherent sheaves on an algebraic variety, of weighted stable maps to quotient stacks $[\mathbb{C}^N/\mu_r]$, and of weighted stable maps to projective varieties.

## 1. Stability conditions on the derived category

It is a classical viewpoint in algebraic geometry (due to Grothendieck) that in order to understand the geometry of an algebraic variety $X$, it is completely sufficient to understand the *category of coherent sheaves* on $X$. Recently, a lot of focus has shifted further to the study of the derived category $D^b(X)$ of the category of coherent sheaves on $X$. It exhibits more symmetries than $X$ itself. Further, two varieties $X \neq Y$ can have isomorphic derived categories $D^b(X) \cong D^b(Y)$, which has been used in many situations to establish connections between the geometry of $X$ and of $Y$.

This naturally leads to the question how to systematically extract geometry from a derived category $D^b(X)$. One approach is via the study of stability conditions on a derived category. This notion was introduced by Bridgeland in [Bri07] as an attempt to mathematically understand the notion of $\pi$-stability of $D$-branes in string theory. One of the two ingredients of a stability condition on $D^b(X)$ is a slicing of $D^b(X)$ into subcategories $\mathcal{P}(\phi)$ indexed by $\phi \in \mathbb{R}$. The objects in $\mathcal{P}(\phi)$ are called "semistable", and are building blocks of $D^b(X)$ in the following sense:

- There are no non-zero maps $A_1 \to A_2$ between semistable objects $A_i \in \mathcal{P}(\phi_i)$ for $\phi_1 > \phi_2$.
- Every object $E$ in $D^b(X)$ can be constructed as an iterated extension of semistable factors: there are semistable objects $A_1, \dots, A_n$ with $A_i \in \mathcal{P}(\phi_i)$ and $\phi_1 > \phi_2 > \cdots > \phi_n$, and

objects $E_2, E_3, \dots, E_n = E$ such that $E_2$ is an extension $A_1 \to E_2 \to A_2$ of $A_1$ and $A_2$, and such that $E_i$ for $i > 2$ is an extension of $E_{i-1}$ and $A_i$.[1]

The other ingredient, called "central charge", allowed Bridgeland to prove that for every triangulated category $\mathcal{D}$, the space of stability conditions on $\mathcal{D}$ is a smooth manifold.

**Polynomial Bridgeland stability conditions.** Stability conditions in the sense of Bridgeland on the derived category $D^b(X)$ are at this point only known for $X$ of dimension 1 or 2 (see [Bri07, Bri03, ABL07]), and for a certain special class of varieties.[2] However, constructions in the string theory literature ([Dou02, AD02, Asp03, AL01]) strongly suggest the existence of stability conditions for higher-dimensional varieties.

In [Bay07], I introduced the notion of polynomial stability conditions on triangulated categories as an attempt to mathematically understand limits ("large-volume limit") of these stability conditions considered by string theorists. Polynomial stability conditions allow greater freedom for the central charge, and yield a finer slicing of $D^b(X)$ indexed by a linearly ordered set $S$ lying above $\mathbb{R}$. In contrast to the small list of examples of Bridgeland stability conditions, we have:

**Theorem 1.1.** [Bay07, Theorem 3.2.2] *There exists a family of polynomial stability conditions on $D^b(X)$ for every normal projective variety $X$.*

I also showed that for surfaces, this family indeed contains "large-volume limits" of Bridgeland stability conditions as constructed in [Bri03, ABL07], in accordance with my original motivation.

**Applications of polynomial stability conditions.** There are various ways in which the polynomial stability conditions on $D^b(X)$ constructed in theorem 1.1 (which we will call "family of standard polynomial stability conditions" for the purpose of this exposition) may contribute to studying the geometry of $X$ in terms of $D^b(X)$. One can study moduli spaces of semistable objects with fixed topological invariants,[3] and how the moduli spaces change when we vary the stability condition ("wall-crossing phenomenon") inside the family of stability conditions.

*How to recover $X$ from $D^b(X)$?* In general, it is not possible to recover $X$ from just $D^b(X)$ alone. A polynomial stability condition may be considered the missing additional data to allow this reconstruction:

**Proposition 1.2.** [Bay07, Proposition 5.1] *The variety $X$ can be recovered from $D^b(X)$ and any standard polynomial stability condition as the moduli space of semistable objects of correct topological invariants.*

While the proposition is rather straightforward to prove in our setting, it makes the promising suggestion to study (birational) modifications of $X$ by studying the wall-crossing for $X$ as a moduli space.

[1] In a triangulated category, an object $E$ is called an extension of objects $A, B$ if there is an exact triangle $A \to E \to B \to A[1]$.

[2] Stability conditions can be constructed in the case when $D^b(X)$ has a complete exceptional collection, see [Mac04].

[3] By topological invariants of an object $E \in D^b(X)$ we denote its class $[E]$ in the "numerical $K$-group" of $D^b(X)$.

*Curve counting invariants.* In contrast to Gromov-Witten theory discussed in the next section, there are two recent theories that approach the classical question of counting the number of curves $C$ inside an algebraic variety $X$ via coherent sheaves. One can think of $C$ as being given by either of the following data:

(1) the ideal sheaf $\mathcal{J}_C$ of a curve, which keeps track of the set of locally defining equations for $C$ inside $X$ at every point of $C$.
(2) the structure sheaf $\mathcal{O}_C$ as a sheaf concentrated on $C$, and coming with a natural generating section $s = 1 \in \mathcal{O}_C$.

While the two viewpoints are equivalent for smoothly embedded curves $C \subset X$, they lead to radically different compactifications of the non-compact moduli space of smoothly embedded curves $C \subset X$: The Hilbert scheme $M_{\mathrm{DT}}$ of ideal sheaves of one-dimensional subschemes, and the moduli space $M_{\mathrm{PT}}$ of "stable pairs" (i.e. pairs $(\mathcal{F}, s)$ of a one-dimensional sheaf $\mathcal{F}$ with a section $s$, satisfying a certain stability assumption).

Thus they also lead to differently defined curve counting invariants, introduced in [MNOP06] and [PT07] respectively.

By viewing both $\mathcal{J}_C$ and stable pairs $(\mathcal{F}, s)$ as objects in the derived category (the latter as the two-term complex $\mathcal{O}_X \to^s \mathcal{F}$), our family of standard polynomial stability conditions can connect the two viewpoints:

**Proposition 1.3.** [Bay07, Proposition 6.1.1] *Let $X$ be a smooth 3-dimensional variety, and let $C \subset X$ be a curve. For a polynomial stability condition $\alpha$, consider the moduli space $M_\alpha$ of objects that are semistable with respect to $\alpha$, and have the same topological invariants as the ideal sheaf $\mathcal{J}_C$ of $C$.*

(1) *There exists an open chamber $\mathcal{C}_{\mathrm{DT}}$ in the family of polynomial stability conditions such that for any $\alpha \in \mathcal{C}_{\mathrm{DT}}$, the moduli space $M_\alpha$ is the moduli space $M_{\mathrm{DT}}$ of ideal sheaves.*
(2) *There exists an adjacent open chamber $\mathcal{C}_{\mathrm{PT}}$ such that for any $\alpha \in \mathcal{C}_{\mathrm{PT}}$, the moduli space $M_\alpha$ is the moduli space $M_{\mathrm{PT}}$ of stable pairs.*

In either case, the semistable objects are exactly the ideal sheaves, respectively stable pairs; so one can see explicitly the change from $M_{\mathrm{DT}}$ to $M_{\mathrm{PT}}$ as a wall-crossing phenomenon in our family of standard polynomial stability conditions.

In [PT07], Pandharipande and Thomas conjecture a relation between the generating functions of DT- and PT-invariants. Proposition 1.3 puts their conjecture into the bigger conjectural framework of wall-crossing formulas for counting invariants in the derived category. This viewpoint originates from the work of Joyce (see [Joy04]), and has been studied by Bridgeland/Toledano-Laredo and Kontsevich/Soibelman (see [BTL08, KS08]).

## 2. Gromov-Witten invariants

In string theory, Gromov-Witten invariants come up as path integrals in sigma models. Mathematically, they count the number of curves (Riemann surfaces) with fixed genus and additional side constraints inside a fixed algebraic variety $X$; slightly more precisely they count the number of "stable" maps from a Riemann surface to $X$ that satisfy the given constraints.

**Stacky Gromov-Witten invariants.** Recently, a lot of interest in Gromov-Witten theory has focused on Gromov-Witten invariants of stacks. Stacks are a notion in algebraic geometry that intrinsically incorporates group actions on varieties, and their Gromov-Witten invariants (defined in

[CR01, AV02, AGV06]) are a way to count curves that takes into account group actions on the target space. Even for simple stacks defined by a linear action of a finite group on $\mathbb{C}^N$, only very few general results are known.

In joint work with C. Cadman (see [BC07]), we developed a new method to study the genus-zero Gromov-Witten invariants of stacks $[\mathbb{C}^N/\mu_r]$ associated to linear group actions of the cyclic group $\mu_r$ of order $r$ on $\mathbb{C}^N$.

The relevant moduli space is the space $\overline{M}_{0,n}(B\mu_r)$ of genus-zero stable maps to the classifying stack $B\mu_r = [\mathrm{pt}/\mu_r]$. It is a compactification of the Hurwitz space of cyclic branched covers of the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. While such a cyclic branched cover is completely determined by the branch points on $\mathbb{P}^1$, in order to compactify one has to allow objects with bigger automorphism groups; the following theorem describes these additional automorphisms along the boundary precisely:

**Theorem 2.1.** [BC07, Theorem 2.5.2] *The moduli space $\overline{M}_{0,n}(B\mu_r)$ can be constructed from the moduli space $\overline{M}_{0,n}$ of stable genus-zero curves with $n$ marked points by an explicitly given sequence of $r$-th root constructions (see* [Cad07]*) along boundary divisors.*

We then introduced a notion of "weighted stable maps". They yield different compactifications of the cyclic Hurwitz spaces, and were motivated by the construction of weighted stable maps to projective varieties discussed in the next section. We studied the wall-crossing behavior under a change of weights, and used it to reduce the computation of Gromov-Witten invariants of $[\mathbb{C}^N/\mu_r]$ to an explicit intersection-theoretic computation on $\overline{M}_{0,n}$ (see [BC07, Theorem 5.1.1]). As a consequence, we obtained new linear recursive relations for Gromov-Witten invariants of $[\mathbb{C}^N/\mu_r]$; for example:

**Proposition 2.2.** [BC07, section 6.3] *The genus-zero Gromov-Witten invariant of $[\mathbb{C}^3/\mu_3]$ for $3n$ marked points can be expressed as a purely combinatorial sum over partitions of $n$.*

**Weighted stable maps to projective varieties.** In [BM06] (joint with Yuri I. Manin) we developed the theory of weighted stable maps from curves to a projective variety. They yield different compactifications of the moduli space than the one introduced by Kontsevich in [KM94] (which is used throughout in Gromov-Witten theory). By introducing virtual fundamental classes, we showed that there is a consistent theory of Gromov-Witten invariants based on weighted stable maps. This exhibits a new algebraic structure on standard (non-weighted) Gromov-Witten invariants.

**Semisimple Quantum Cohomology.** In [Bay04], I studied the behavior of "semisimplicity of Quantum Cohomology". This is a property of genus zero Gromov-Witten invariants of a projective variety $X$. Among other things, it implies that all Gromov-Witten invariants of $X$ (including higher genus) are completely determined by a finite number of genus-zero Gromov-Witten invariants.

**Theorem 2.3.** [Bay04, Theorem 3.1.1] *If $X$ has semisimple quantum cohomology, then the same holds true for for the blow–up of $X$ at any number of points.*

This gave further evidence for a conjecture by Dubrovin (and suggested a slight modification of the conjecture).

## 3. Further directions of research

**Stability conditions for threefolds.** In a joint project with A. Bertram and G. Todorov, we are studying stability conditions for three-dimensional varieties. We have made partial progress in using the polynomial stability conditions of [Bay07] for constructing stability conditions in the sense of Bridgeland. These promise to have an even richer structure of wall-crossing phenomena.

**Local wall-crossing phenomena.** Due to a celebrated result ([BKR01]), the derived category of the singular quotient $X = [\mathbb{C}^3/G]$ (where $G$ is a finite subgroup of $\mathrm{SL}_3(\mathbb{C})$) is isomorphic to the derived category of a smooth three-dimensional variety $Y$ lying above the quotient. In joint work with Y. Jiang, we are comparing stability conditions on $D^b(X) \cong D^b(Y)$ constructed from $X$ and $Y$, respectively, in order to compare counting invariants on either side.

**Stacky higher-genus invariants.** In a current project with C. Cadman and R. Cavalieri, we are using the Grothendieck-Riemann-Roch theorem for stacks to study higher-genus Gromov-Witten invariants of $[\mathbb{C}^3/\mu_3]$.

**Higher-genus invariants of Grassmannians.** In a project with B. Kim, I. Ciocan-Fontanine and Y.-P. Lee, we are studying higher-genus invariants of Grassmannians. Our approach would extend the earlier work in [BCFK04, CFKS06], which relates the genus-zero Gromov-Witten invariants of the Grassmannian $G(r, n+1)$ of $r$-planes in $\mathbb{C}^{n+1}$ to the genus-zero Gromov-Witten invariants of the $r$-fold product $(\mathbb{P}^n)^r$ of projective space.

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