Solutions to Quiz no. 5-Remake

November 13, 2006

1. Determine the value of the following series:

$$5 + 2 + \frac{4}{5} + \frac{8}{25} + \frac{16}{125} + \cdots =$$

Solution: To go from one term to the next, you have to multiply by $\frac{2}{5}$. So this is a geometric series with $a = 5$ and $r = \frac{2}{5}$, and the series hast the value

$$\frac{a}{1-r} = \frac{5}{1-\frac{2}{5}} = \frac{25}{3}$$

2. Find out which of the following series converges. Indicate which test you use.

(a) $\sum_{k=2}^{\infty} \frac{1}{k \cdot (\ln k)^3}$

Solution: Use the integral test. Since $f(x) = \frac{1}{x(\ln x)^3}$ is decreasing, the sum converges if and only if the integral $\int_2^{\infty} \frac{1}{x(\ln x)^3} dx$ converges. With $u = \ln x$, $du = \frac{1}{x} dx$ we get

$$\int \frac{1}{x(\ln x)^3} dx = \int \frac{1}{u^3} du = -\frac{1}{2u^2} + C = -\frac{1}{2(\ln x)^2} + C$$
$$\int_2^{\infty} \frac{1}{x(\ln x)^3} dx = \lim_{a \to +\infty} \left[-\frac{1}{2(\ln x)^2} \right]_2^a$$
$$= \lim_{a \to +\infty} -\frac{1}{2(\ln a)^2} + \frac{1}{2(\ln 2)^2} = \frac{1}{2(\ln 2)^2}$$

So the integral converges, and the sum converges, too.

Popular mistake: Limit comparison with the harmonic series gives nothing: $\lim \frac{\frac{1}{k(\ln k)^3}}{\frac{1}{k}} = 0$, but as $\sum \frac{1}{k}$ diverges, this doesn't tell us anything.

(b) $\sum_{n=2}^{\infty} \frac{\sqrt{n^2+1}}{\sqrt{n}(n^3+1)}$

Solution: We use the limit comparison test:

$$\frac{\sqrt{n^2+1}}{\sqrt{n}(n^3+1)} \approx \frac{\sqrt{n^2}}{\sqrt{n}n^3} = \frac{1}{n^{\frac{5}{2}}},$$

so we set $b_n = \frac{1}{n^{\frac{5}{2}}}$. Then $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n^{\frac{5}{2}}}$ converges, as

$\frac{5}{2} > 1$. The limit comparison is

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n^2+1}}{\sqrt{n}(n^3+1)}}{\frac{1}{n^{\frac{5}{2}}}} = \lim_{n\to\infty} \frac{\sqrt{n^2+1}\sqrt{n}n^2}{\sqrt{n}(n^3+1)} = \lim_{n\to\infty} \frac{\sqrt{n^2+1}n^2}{(n^3+1)}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}\sqrt{n^2+1}}{1+\frac{1}{n}} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n^2}}}{1+\frac{1}{n}} = 1,$$

so the original series converges, too.

(c) $\sum_{k=1}^{\infty} \frac{k^2 2^k}{3^k}$

Solution: Use the ratio test:

$$\rho = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{(k+1)^2 2^{k+1}}{3^{k+1}}}{\frac{k^2 2^k}{3^k}} = \lim_{k\to\infty} \frac{(k+1)^2}{k^2}\frac{2^{k+1}}{2^k}\frac{3^k}{3^{k+1}}$$

$$= \lim_{k\to\infty} \frac{(k+1)^2}{k^2}\frac{2\cdot 2^k}{2^k}\frac{3^k}{3\cdot 3^k} = \frac{2}{3}\lim_{k\to\infty}\frac{(k+1)^2}{k^2}$$

$$= \frac{2}{3}\left(\lim_{k\to\infty}\frac{k+1}{k}\right)^2 = \frac{2}{3}\left(\lim_{k\to\infty}\frac{1+\frac{1}{k}}{1}\right)^2 = \frac{2}{3}$$

Since $\rho < 1$, the series converges.

3. $\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)}$

Solution: **(Sketch)** Partial fraction decomposition gives $\frac{1}{(k+1)(k+2)} = \frac{1}{k+1} - \frac{1}{k+2}$. So $S_n = \sum_{k=1}^{n} \frac{1}{(k+1)(k+2)} = \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{n+1}-\frac{1}{n+2}\right) = \frac{1}{2} - \frac{1}{n+2}$ and $\sum_{k=1}^{\infty}\frac{1}{(k+1)(k+2)} = \lim_{n\to\infty} S_n = \frac{1}{2}$.

4. Assume you know that the sequence given by

$$a_1 = 1, \quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$

has a limit $L = \lim_{n\to\infty} a_n$. Determine the limit by finding an equation for L.

Solution: Since $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$ is true for all n, it is also true that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$. Now $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$. So we get

$$L = \frac{1}{2}\left(L + \frac{2}{L}\right) \qquad 2L = L + \frac{2}{L}$$

$$L = \frac{2}{L} \qquad L^2 = 2$$

$$L = \pm\sqrt{2}$$

Since all a_n are positiv, the limit must be $\sqrt{2}$.

Solutions for quiz no. 6

November 14, 2006

1. Which of the following series converge? Which of them converge absolutely?

   (a) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

   Solution: Let $a_n = \frac{1}{n \ln n}$. Then $a_2 > a_3 > a_4 > \dots$, and the series above is the alternating series $a_2 - a_3 + a_4 - a_5 \pm \dots$. Hence it converges.

   Does it converge absolutely, i.e. does $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converge? Use the integral test: $f(x) = \frac{1}{x \ln x}$ is decreasing, so the sum converges if and only if the integral $\int_2^{\infty} \frac{1}{x \ln x} dx$ converges. With $u = \ln x$ we see that $\int \frac{1}{x \ln x} dx = \frac{1}{u} du = \ln u + C = \ln(\ln x) + C$, so

   $$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{a \to \infty} \int_2^a \frac{1}{x \ln x} dx = \lim_{a \to \infty} (\ln(\ln a) - \ln(\ln 2)) = +\infty,$$

   so the integral diverges, and our original sum does not converge absolutely.

   Popular mistakes: Since neither powers nor factorials are involved, you shouldn't use the ratio test. If you try it, you get (for the absolute ratio test):

   $$\rho = \lim_{n \to \infty} \frac{\left|\frac{(-1)^{n+1}}{(n+1)\ln(n+1)}\right|}{\left|\frac{(-1)^n}{n \ln n}\right|} = \lim_{n \to \infty} \left|\frac{n \ln n}{(n+1)\ln(n+1)}\right|$$

   $$= \lim_{x \to \infty} \frac{x \ln x}{(x+1)\ln(x+1)} =^L \lim_{x \to \infty} \frac{\ln x + 1}{\ln(x+1) + 1} =^L \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{x+1}} = 1,$$

   but **for $\rho = 1$ the ratio test is inconclusive!**

   (b)

   $$\frac{1}{1} + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} + \frac{1}{128} - \frac{1}{256} =$$

   Solution: With $a_0 = \frac{1}{1}$, $a_1 = \frac{1}{2}, a_2 = -\frac{1}{4}$ etc., we have $|a_n| = \frac{1}{2^n}$. So

   $$\sum_{n=0}^{\infty} |a_n| = \frac{1}{1 - \frac{1}{2}} = 2,$$

   so the series converges absolutely. Hence the series converges.

2. Find the set of convergence for the following power series:

(a) $\sum_{n=0}^{\infty} \frac{1}{n! \cdot n} x^n$

Solution: Use the absolute ratio test:

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{|\frac{x^{n+1}}{(n+1)!\cdot(n+1)}|}{|\frac{x^n}{n!\cdot n}|} = \lim_{n\to\infty} |\frac{x^{n+1}}{x^n} \frac{n!}{(n+1)!} \frac{n}{n+1}| \\
&= \lim_{n\to\infty} |\frac{x^n \cdot x^1}{x^n} \frac{n!}{(n+1)\cdot n!} \frac{n}{n+1}| = |x| \lim_{n\to\infty} \frac{n}{(n+1)^2} \\
&= |x| \lim_{n\to\infty} \frac{n}{n^2+2n+1} = |x| \lim_{n\to\infty} \frac{1}{n+2+\frac{1}{n}} = 0
\end{aligned}$$

Hence the power series converges for **all** x (since $\rho < 1$ for all x).

(b) $\sum_{n=0}^{\infty} \frac{(x-4)^n}{n^3}$

Solution: Again, start with the absolute ratio test:

$$\begin{aligned}
\rho &= \lim_{n\to\infty} |\frac{\frac{(x-4)^{n+1}}{(n+1)^3}}{\frac{(x-4)^n}{n^3}}| = \lim_{n\to\infty} |\frac{(x-4)^{n+1}}{(x-4)^n} \frac{(n+1)^3}{n^3}| \\
&= |x-4| \lim_{n\to\infty} \frac{(n+1)^3}{n^3} = |x-4| \left(\lim_{n\to\infty} \frac{n+1}{n} \right)^3 = |x-4|
\end{aligned}$$

So when is $\rho < 1$? This is the case when $-1 < x - 4 < 1$, i.e. when $3 < x < 5$.

Now we need to consider the endpoints:

$x = 5$ We get $\sum_{n=0}^{\infty} \frac{(5-4)^n}{n^3} = \sum_{n=0}^{\infty} \frac{1}{n^3}$ which is a convergent p-series, as $3 > 1$.

$x = 3$ We get $\sum_{n=0}^{\infty} \frac{(3-4)^n}{n^3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^3}$. This converges, and you can pick among two possible reasonings:

- This is an alternating series with $a_n = \frac{1}{n^3}$, which satisfies $a_n > a_{n+1}$ and $\lim_{n\to\infty} a_n = 0$, so it converges.
- If we take the absolute values, we get back the convergent series $\sum \frac{1}{n^3}$ from the previous case $x = 5$, so the series converges absolutely, and so it converges.

So the convergence set is $[3, 5]$.