Solutions Midterm no. 3 (1220-5, Fall 2006)

The following are mostly solutions sketches, and some remarks on common mistakes. Don't hesitate to bug me if you would like to see more details on some of the problems.

1. (8 points) Compute the **value** of the folloing series:

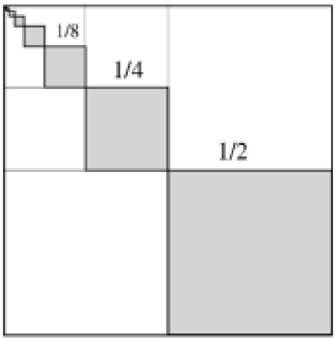
   (a) $\frac{1}{2} + \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \frac{16}{243} + \ldots$
   Solution: This is a geometric series with $a = \frac{1}{2}$ and $r = \frac{2}{3}$. □

   (b) *Hint: Find a formula for* S_n. $\sum_{k=2}^{\infty} \frac{1}{k^2} - \frac{1}{(k+1)^2}$
   Solution:

   $$S_n = \left(\frac{1}{4} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{16}\right) + \cdots + \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) = \frac{1}{4} - \frac{1}{(n+1)^2}$$

   so $\lim S_n = \frac{1}{4}$. □

2. (6 points) A square with side length of 1 is divided into 4 smaller squares of side length $\frac{1}{2}$, and the lower right square is shaded. Then this is repeated with the upper left square, and so on, as in figure 1. Find the total area of the shaded region.



Solution: The side lengths of the shades squares are $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ so its areas are $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots$. So we get $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots$ which is the geometric series with $a = \frac{1}{4}$ and $r = \frac{1}{4}$. □

3. (8 points) Which of the following two series converges? Indicate which test you use.

   (a) $\sum_{n=1}^{\infty} \frac{n^2+n+1}{3^n}$ **Solution:** Use the ratio test. □

   (b) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ **Solution:** Use the integral test. **Don't** use the ratio test unless you have an exponential or factorial in the denominator! □

4. (6 points) Does the following series converge? Does it converge absolutely? $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n+1}}{\sqrt{n^3+n^2}}$

   Solution: This problem has two questions: Does $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n+1}}{\sqrt{n^3+n^2}}$ converge? Does $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{\sqrt{n^3+n^2}}$ converge (absolute convergence of the original series)?

   For the second part: **Don't** use the ratio test here! Instead, you need to use the limit comparison theorem. By just dropping all terms with lower powers of n, we get $b_n = \frac{\sqrt{n}}{\sqrt{n^3}} = \sqrt{\frac{n}{n^3}} = \frac{1}{n}$, so $\sum_n b_n$ diverges. The limit comparison test will then show that $\sum_n |a_n|$ diverges, too. □

5. (8 points) For each of the following **sequences**, determine whether it converges, and compute the value of the limit if it does.

   (a) $a_n = \frac{n^3}{3^n}$

   Solution: Use $\lim_{n\to\infty} \frac{n^3}{3^n} = \lim_{x\to\infty} \frac{x^3}{3^x}$. The latter can be easily computed using the rule of L'Hopital. Note that the derivative of 3^x is $\ln 3 \cdot 3^x$, **not** $3 \cdot 3^{x-1}$ or $x \cdot 3^{x-1}$.

   You shouldn't use the ratio test here, which would be a test for the **series** $\sum_{n=0}^{\infty} a_n$, while the problem just asks about the **sequence** a_n, i.e. about $\lim_{n\to\infty} \frac{n^3}{3^n}$. □

   (b) *Hint: Use the the squeeze theorem.* $a_n = \frac{\sin n}{n \cdot \sqrt{n}}$

   Solution: Use $-1 \leq \sin n \leq +1$. □

6. (4 points) The sequence (a_n) is defined by $a_1 = 1, a_{n+1} = \frac{2}{a_n} + 1$. It can be shown that this sequence converges. Determine its limit.

   Solution: Set $L = \lim_{n\to\infty} a_n$. Since $a_{n+1} = \frac{2}{a_n} + 1$ holds for all n, we also have $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{2}{a_n} + 1$, which gives $L = \frac{2}{L} + 1$. Then:

$$
\begin{aligned}
L &= \frac{2}{L} + 1 & \quad | \cdot L \\
L^2 &= 2 + L & \quad | -2 - L \\
L^2 - L - 2 &= 0.
\end{aligned}
$$

   The solutions or $L = 2$ and $L = -1$. Since all a_n are positive, the limit must be 2. □

7. (6 points) Find the convergence set of the following power series: $p(x) = \sum_{n=0}^{\infty} \frac{(x+3)^n}{n^2}$

   Solution: Use the absolute ratio test. Don't forget to use absolute values! You will get $\rho = |x+3|$. So $\rho < 1$ holds when $-1 < x + 3 < 1$, i.e. for the interval $(-4,-2)$. Checking the endpoints $x=-4$ and $x=-2$ will show that the convergence set is $[-4,-2]$. $\square$

8. (8 points) Find the taylor series of $f(x)=\cosh x$ by computing its derivatives. Give a formula for the n-th term of the power series, and determine its convergence set.

   Solution:

$$\begin{aligned} f^{(0)}(x) &= \cosh x \\ f^{(1)}(x) &= \sinh x \\ f^{(2)}(x) &= \cosh x \\ f^{(3)}(x) &= \sinh x \end{aligned}$$

   Since $\cosh 0 = 1$ and $\sinh 0 = 0$ one gets, using the Taylor series formula,

$$f(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

   Be careful when plugging in $n+1$ when using the ratio test:

$$\begin{aligned} \rho &= \lim_{n\to\infty} \left|\frac{x^{2(n+1)}}{(2(n+1))!}\right| / \left|\frac{x^{2n}}{(2n)!}\right| = \lim_{n\to\infty} \left|\frac{x^{2n+2}}{x^{2n}} \cdot \frac{(2n)!}{(2n+2)\cdot(2n+1)\cdot(2n)!}\right| \\ &= \left|x^2\right| \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = \left|x^2\right| \cdot 0 = 0 \end{aligned}$$

   so the power series converges for all x. $\square$

9. (10 points) Find a power series for each of the following functions.

   (a) $\frac{x}{1+x^7}$ **Solution:** This is the geometric series with $a = x$ and $r = -x^7$, so $a + ar + ar^2 + \cdots = x - x^8 + x^{15} - x^{22} \pm \ldots$. $\square$

   (b) $(1+x)\ln(1+x)$
   Solution:

$$(1+x)\big(x - \frac{x}{2} + \frac{x^3}{3} - \frac{x^4}{4} \pm \ldots\big) = x + \big(1-\frac{1}{2}\big)x^2 + \big(-\frac{1}{2}+\frac{1}{3}\big)x^3 + \big(\frac{1}{3}-\frac{1}{4}\big)x^4 + \ldots$$

   $\square$

   (c) $\int_0^x e^{t^4}\,dt$ **Solution:** From the power series of e^x one gets $e^{x^4} = 1 + \frac{x^4}{1!} + \frac{x^8}{2!} + \frac{x^{12}}{3!} + \ldots$, of which we need to take the anti-derivative. So we get $x + \frac{x^5}{5\cdot 1!} + \frac{x^9}{9\cdot 2!} + \frac{x^{13}}{13\cdot 3!} + \ldots$. $\square$