There and Back Again - The Applied Math Journey

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What is applied math?

Applied math can be thought of as understanding engineering and physical problems at a very basic (although complex!) level. Understanding a mathematical model gives us great insight into our reality.

How does one 'apply' math?

- *(there)* First the physical problem, such as understanding how an object behaves when it is dropped from a certain height, or put in an electromagnetic field, must be 'mathematized'- that is described mathematically.
  - Physical laws, such as Newton's Second Law or Maxwell's Equations help describe or model forces and relationships between things like electricity and magnetism.

Newton's Second Law

$$\mathbf{F} = m\mathbf{a}$$

Maxwell's Equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = \frac{-\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

What does the math say about our world?

- Next mathematicians use relationships between objects to try and understand a system as a whole.
  - Symmetry is widely used throughout mathematics to simplify understanding of things
- *(and back again)* Now we study the mathematical problem and apply the insight gained there to our reality.
  - Combining intuition (what should happen) and mathematical rigor (what does happen mathematically) we study the underlying problem.

Undergraduate Studies

At an undergraduate level, mathematicians all study very similar subjects which build up a core intuition for further research.

- Calculus - *this is where math gets interesting!*
- Linear Algebra - general understanding of mathematical relationships
- Differential Equations - computational understanding of some engineering-type problems
- Analysis - deeper understanding of mathematical relationships
- Numerics - the art of staring at a computer for way too long, more fun than it sounds

Undergraduate Studies

Studying these subjects can draw all sorts of cool pictures in your mind!

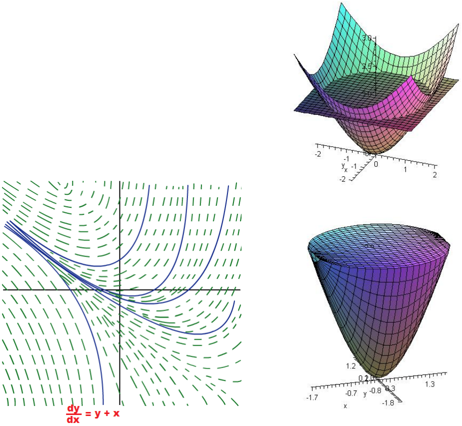


Figure: Images from Differential Equations and Calculus III courses.

Graduate Studies

As a beginning graduate student, we choose between pure and applied mathematics. We begin to develop our skills toward studying physical problems or models derived directly from reality. Applied mathematicians spend the first few years studying these subjects in much greater depth and detail.

- Differential Equations - understanding of mathematical objects and operators (derivatives, integrals, etc.)
- Numerical Analysis - computational/scientific code and the theory
- Analysis - deep theoretical knowledge of mathematical laws and objects
- Physical or Biological application classes - depending on specialization

Graduate Studies

Numerical Solution to the PDE

$$\begin{aligned} \Delta u(x,y) &= f(x,y) \quad \text{in } \Omega \\ u(x,y) &= g(x,y) \quad \text{on } \partial\Omega \end{aligned}$$

Specialization

Now that we have spent some time developing skills and mathematical intuition, it is time to specialize. We generally try to pick areas of current research which interest us most, and spend some time getting caught up in that particular field.

Numerical Analysis

Specialization in Numerical Analysis may have you looking at numerical algorithms or code such as Newton's Method.

$$\begin{cases} f = @(2x^2 - 4) \\ fp = @(4x) \\ x_0 = 0 \\ \text{for } k = 0 \text{ to } max \\ \qquad x_{k+1} = x_k - f(x_k)/fp(x_k) \\ \text{end} \end{cases}$$

A numerical analyst studies the convergence and properties of routines much more complicated than this. These routines have applications throughout the realm of applied mathematics, including modeling and differential equations.

Functional Analysis/PDE Theory

Specialization may include imaging techniques involved with inverse problems, such as the Radon and X-Ray Transform.

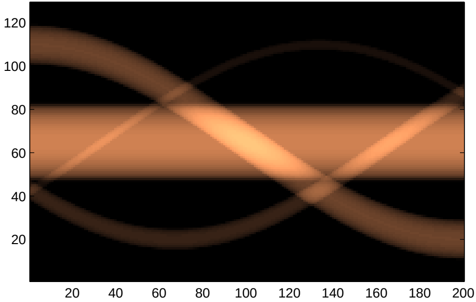
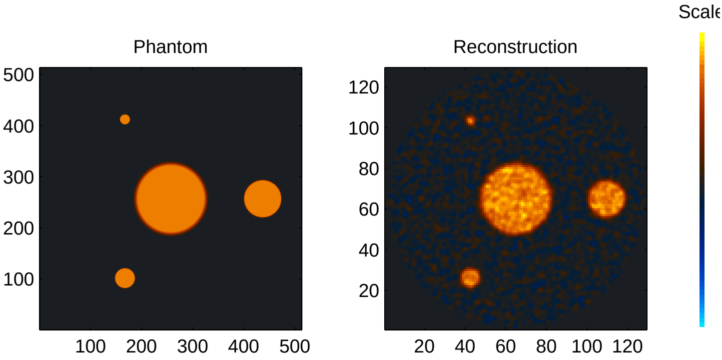


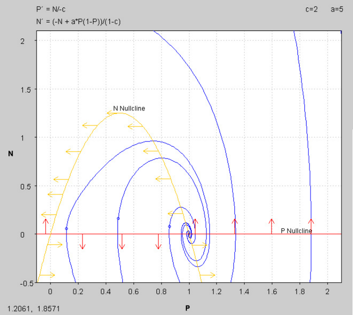
Figure: Radon Transform of a relatively simple image

Functional Analysis/PDE Theory

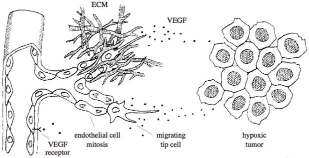
Recovering even a simple image such as this is not a very easy problem!



Specializing generally involves modeling and interpreting physical and biological processes. Cancer research is a huge topic studied by math biologists, as well as growth of other living organisms.



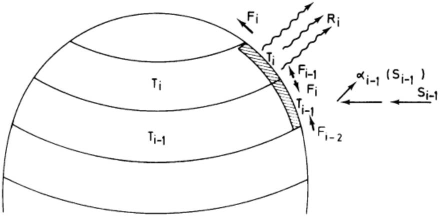
(a) Phase plane analysis for growth of fungi



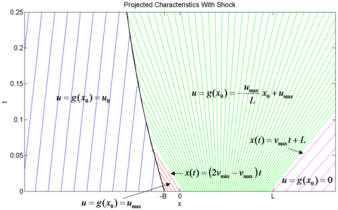
(b) Image of angiogenesis

Modeling

This branch of mathematics focuses on modeling physical or real world phenomena. Many climate models have been created modeling wind currents, ocean currents, etc.



(c) Model of sunlight reflection and absorbtion



(d) Conservation Law and Method of Characteristics

My Areas of Research

Throughout my graduate career, I have studied

- Modeling - Snowbird traffic flow
- Modeling/Numerics - Polycrystaline grain growth
- PDE Theory and Inverse Problems - Inverse Born series and internal measurements

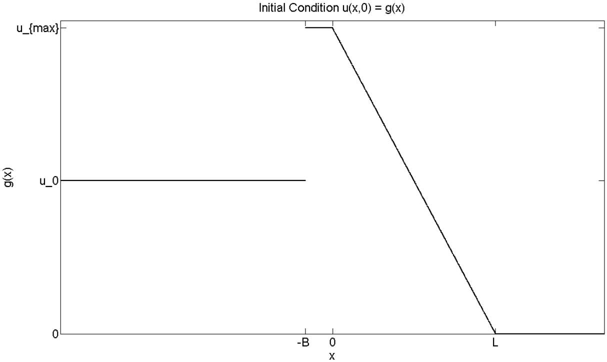
Modeling

Modeling the Red Snake with Andy Thaler

Modeling

Using basic Partial Differential Equation theory, we observed a traffic model which described the flow of traffic from Snowbird, Ut on a busy winter afternoon.

$$u_t + (2mu + v_{max})u_x = 0 \text{ in } \mathbb{R} \times (0, \infty)$$
$$u = g(x) \text{ on } \mathbb{R} \times \{0\}$$



Modeling

The solution or density profile of cars at Snowbird over time

Grain Growth

Polycrystaline Grain Growth with Yekaterina Epshteyn

Grain Growth

Given an initial system of grains which grow and shrink according to certain laws, can we predict an overall 'macroscopic' behavior?

$$\overbrace{\rightarrow\qquad\qquad\qquad\leftarrow}^{v_i}$$

$$|\underbrace{\underline{\qquad\qquad\qquad}}_{\alpha_{i-1}}|\underbrace{\underline{\qquad\qquad\qquad}}_{\alpha_i}|\underbrace{\underline{\qquad\qquad\qquad}}_{\alpha_{i+1}}$$

$$\phi(\alpha) = 1 + \frac{1}{2}\sin(2\alpha)^2$$

where the velocity of growth of the grain i is given by

$$v_i = \phi(\alpha_{i-1}) - 2\phi(\alpha_i) + \phi(\alpha_{i+1})$$

Grain Growth

This lead us to model the system in a probabalistic sense. Specifically we obtained a probability density kinetic law given by

$$\frac{\partial \rho(\alpha, \alpha_l, \alpha_r, t)}{\partial t} = \rho(\alpha, \alpha_l, \alpha_r, t) v(\alpha, \alpha_l, \alpha_r) + Flux(\alpha, \alpha_l, \alpha_r, t)$$

Grain Growth

Ultimately we achieved a model which accurately describes the behavior of such a large scale system.

Sparse Internal Measurements and the Inverse Born Series with Fernando Guevara-Vasquez

Inverse Problems and Imaging

The Schrödinger Equation given below can be used to model wave propagation through a medium. This equation and others have applications in geothermal imaging (oil reservoirs), as well as medical imaging (Ultrasound Modulated Electrical Impedence Tomography).

$$\begin{aligned} \Delta u_i + q u_i &= \phi_i \quad \text{in } \Omega \\ u_i &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Using a series of internal measurements (physically they correspond to average pressure readings - hydraulic, acoustic, etc) of the form

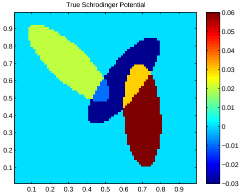
$$\langle \phi_j, u_i - u_{0,i} \rangle_{L^2(\Omega)} = \int_\Omega \phi_j (u_i - u_{0,i}) dx$$

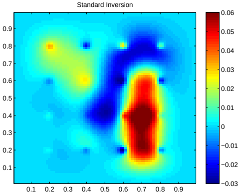
we try to invert the above equation. That is we try to recover the coefficent q. This can be thought of as

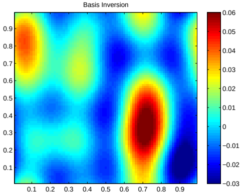
$$``q = \frac{\phi_i - \Delta u_i}{u_i}"$$

Inverse Problems and Imaging

Using what is known as the inverse Born series, we arrive at what is known as a nonlinear inversion technique to recover q. As seen in the images below, there is much to be studied with such an inverse problem!

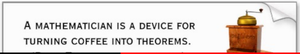






A Day in the Life

- Begin the day with a pot of coffee and remind yourself what you were doing the day before.
- Stare at the wall and hope that you break through the barrier you have been hitting mentally (not the wall).
- Test the idea with a quick piece of code (typically Maple or Matlab).
- Make another pot of coffee.
- Try to prove a result based on the idea you had earlier (convergence, stability, existence, etc.)
- Attend classes or meet with advisors and continue working on results which you believe (intuition) should be attainable.
- Attend seminars and colloquia.



Conclusion

Applied mathematics is a very fun area of research! We have the privledge to study unsolved physical problems, create new and beautiful ideas, and try to apply abstract mathematical theory to the world we all live in. Plus it gives us an excuse to drink coffee...

References and Collaborators

- Fernando Guevara-Vasquez
- Yekaterina Epshteyn
- Andy Thaler
- RTG IPDE 2012 Summer Workshop - Code courtesy of Leonid Kunyansky
- Angiogenesis image courtesy of Jackson Cancer Modeling Group - University of Michigan
- Sunlight radiation image courtesy of Penn State University - Department of Meteorology