“When am I ever going to use this?”

Patrick Bardsley

University of Utah

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Overview

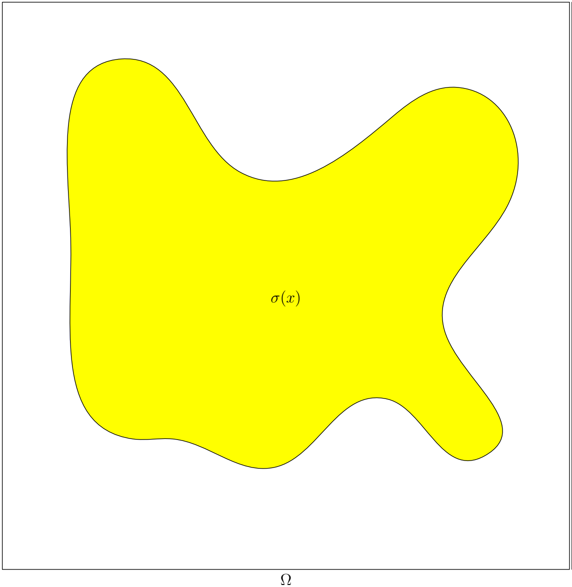
- Hydraulic tomography (underground reservoir imaging) and the inverse Born series. Joint work with F. Guevara Vasquez.[1]
- Polycrystalline materials (plastics) and understanding grain growth processes. Joint work with Y. Epshteyn.
- Imaging with noise - (stealth imaging!) Joint work with F. Guevara Vasquez.

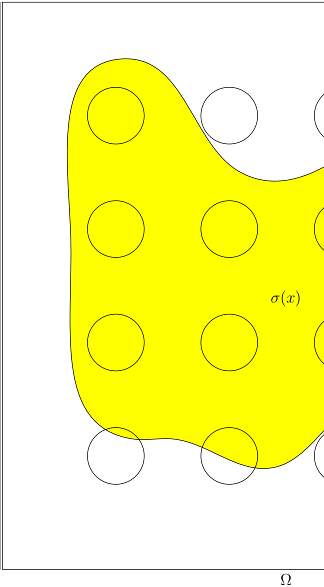
Hydraulic Tomography

Hydraulic Tomography: Imaging or observing an underground reservoir or aquifer using indirect measurements.

Uses: Oil prospection, natural gas prospection, fracking, and hopefully some environmentally friendly situations too!

Hydraulic tomography





Hydraulic Tomography

Partial differential equations give us a connection between what we can measure and what we would like to know. In this case

Physical law given by PDE:

$$\begin{aligned} S\frac{\partial u_j}{\partial t} &= \nabla \cdot (\sigma \nabla u_j) - \phi_j \qquad &\text{for } \mathbf{x} \in \Omega, t \in \mathbb{R} \\ u_j(\mathbf{x}, 0) &= \text{Initial Condition} \qquad &\text{for } \mathbf{x} \in \Omega. \end{aligned}$$

Measurement:

$$M_{i,j}(t) = \int_\Omega \phi_j(\mathbf{x}, t) *_t u_i(\mathbf{x}, t) d\mathbf{x}$$

Unknown: $S(x, y, z)$ and $\sigma(x, y, z)$.

Hydraulic Tomography

PDE:

$$S\frac{\partial u}{\partial t} = \nabla \cdot (\sigma \nabla u_i) - \phi_i \qquad \text{for } \mathbf{x} \in \Omega, t \in \mathbb{R}$$

This actually means

$$\begin{aligned} S(x,y,z)\frac{\partial u(x,y,z,t)}{\partial t} =& \partial_x \left(\sigma(x,y,z)\partial_x u(x,y,z,t)\right) \\ &+ \partial_y \left(\sigma(x,y,z)\partial_y u(x,y,z,t)\right) \\ &+ \partial_z \left(\sigma(x,y,z)\partial_z u(x,y,z,t)\right) - \phi(x,y,z) \\ &\text{for } (x,y,z) \in \Omega, t \in \mathbb{R}. \end{aligned}$$

Gross right? That's why we use the triangles ∇.

Hydraulic Tomography

PDE:

$$S(\partial_t u_i) = \nabla \cdot (\sigma \nabla u_i) - \phi_i$$

So our goal is to determine the functions $S(x, y, z)$ and $\sigma(x, y, z)$ from the measurements

$$M_{i,j}(t) = \int_\Omega \phi_j(\mathbf{x}, t) *_t u_i(\mathbf{x}, t) d\mathbf{x}$$

with the physical law given by the PDE (which connects u_i with the unknown functions.)

How?

The Born Series

Remember your favorite topic... series? Here is a real world application of them!

- We have some data $\mathbf{d}$ (those measurements $M_{i,j}$).
- We have some unknown function information $\mathbf{h}$ (info about S and σ).
- We can prove the data $\mathbf{d}$ is related to the unknown information $\mathbf{h}$ through a series called the forward Born series:

$$\mathbf{d} = \sum_{n=1}^{\infty} a_n(\mathbf{h}^{\otimes n}) \quad \text{where} \quad a_n \sim \frac{f^{(n)}(\mathbf{0})}{n!}.$$

Think of this as a Taylor series of functions.

The Inverse Born Series

To get a reconstruction, we guess that the function information **h** can be built from a series of the data **d** in the form

$$\mathbf{h} = \sum_{n=1}^{\infty} b_n(\mathbf{d}^{\otimes n}).$$

where we know what the b_n's are. In the real world:

$$\mathbf{h} \approx \sum_{n=1}^{N} b_n(\mathbf{d}^{\otimes n}),$$

much like a Taylor polynomial of degree N.

Hydraulic Tomography

This method of reconstruction works to some extent:

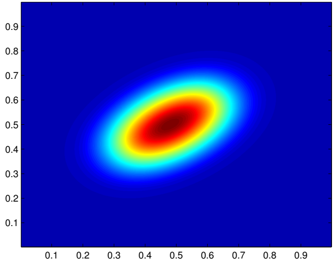
- We know the error we make during reconstruction.
- Only works for certain function information **h** and data sets **d**. In particular the "size" of the function information $\|\mathbf{h}\| \leq 1/\mu$ where μ is a known number.
- The smallness condition comes from

$$\sum_{n=1}^{\infty} r^n = \frac{1}{1-r}$$

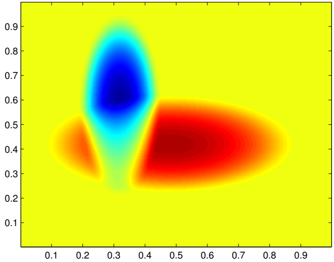
if and only if $|r| < 1$: Geometric Series.

Hydraulic Tomography True Functions

Conductivity $\sigma - 1$



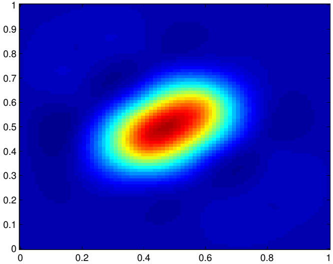
Storage Coefficient S



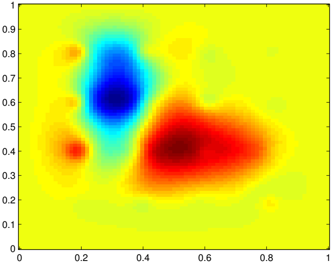
Check it out... contour plots... remember them?

Hydraulic Tomography Reconstructions

Conductivity $\sigma - 1$



Storage Coefficient S



Reconstructions using inverse Born series of order 5 (much like an inverse Taylor polynomial of degree 5).

Hydraulic Tomography

Using this method we can recover an estimate of how large an aquifer is, and estimate how much of a certain substance (gas, oil, water) it contains.

Tools from Math 1320:

- Partial derivatives
- Integrals
- Series (Taylor series ideas)
- Geometric series
- Contour plots
- Multivariable functions $u(x, y, z, t), \sigma(x, y, z)$, etc.

Polycrystalline Grain Growth

Materials science: Understanding growth and formation of certain types of polymers, plastics, polycrystalline materials from a mathematical perspective.

Uses: Predicting material properties such as shear/tensile strength from understanding how the material forms.

Polycrystalline Grain Growth

Materials such as plastics have a microscopic polycrystalline grain structure. During formation these grains grow and shrink according to known empirical laws i.e. big grains eat little grains.

http://www.youtube.com/watch?v=J_2FdkRqmCA

Polycrystalline Grain Growth

Another empirical law says that grains grow according to how their molecular lattice structures are aligned.

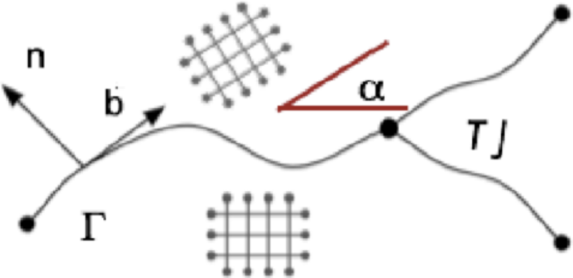


Image source: Barmak et al. [2]

Polycrystalline Grain Growth

People (one of my project advisors for example) have developed theory which describes how the grains grow in a probabalistic sense.

- Grain structures with a favorable lattice orientation grow.
- Grain structures with an unfavorable lattice orientation shrink.
- The probability of finding a particular grain structure lattice orientation $\rho(\alpha, t)$ changes in a predictable way over time.

$$\partial_t \rho = \partial_\alpha \left(\lambda \partial_\alpha \rho + \rho \psi'\right)$$

where ψ measures the favorability of a particular lattice orientation.

Polycrystalline Grain Growth

The location of a "particle" represents the particular lattice orientation at that location. This lattice orientation changes (so the particle changes location) as the polymer develops in time.

Polycrystalline Grain Growth

The probability of a particular lattice orientation as time evolves.

Polycrystalline Grain Growth

Benefit: knowing which lattice structures are more likely than others, we can predict how the material develops over time. This allows us to predict material properties such as strength, weight, etc.

Tools from Math 1320:

- Derivatives, partial derivatives
- Probability - requires integrals to use effectively
- Multivariable functions $\rho(\alpha, t)$

Spoiler Alert!

The Maclaurin series of $x\sin(x)$ looks like this:

$$x\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right)=x^2-\frac{x^4}{3!}+\frac{x^6}{5!}-\frac{x^8}{7!}+\cdots$$

I wonder what the Maclaurin series are for $x\cos(x)$ and xe^x?

Imaging With Noise

Imaging with noise: Using noisy sources (random signals such as white noise) we hope to image objects in a medium using recorded echoes and data processing.

Uses: Developing more efficient medical imaging devices, earthquake detection, seismic imaging, stealth applications.

Array Imaging

The setup: Send out a signal - wait and record echoes - interpret location of reflector based on when echoes are received.

Image source: Borcea et al. [3]

The Math Behind Imaging: Waves

Mathematically we describe movement of signal (sound signal) by how the acoustic pressure $u(x, y, z, t)$ at a location (x, y, z) changes over time t.

The Wave Equation:

$$\frac{1}{c^2}\partial_{tt}u - \partial_{xx}u - \partial_{yy}u - \partial_{zz}u = F$$

$$\frac{1}{c^2}\partial_{tt}u - \Delta u = F$$

where c is the speed of (sound) waves in the medium (bodily tissue), and F is the source of waves (oscillations of ultrasound transducer).

Measurements

Acoustic pressure u at array without reflector:

$$u(\mathbf{x}_r, t) = \frac{1}{2\pi} \int \hat{G}(\mathbf{x}_r, \mathbf{x}_s, \omega) \hat{F}(\omega) e^{i\omega t} d\omega$$

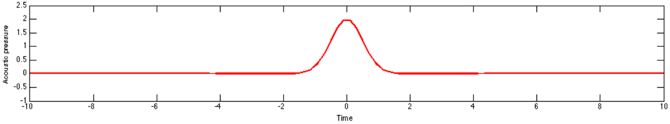
Acoustic pressure u at array with reflector with intensity ρ:

$$\begin{aligned} u(\mathbf{x}_r, t) \approx & \frac{1}{2\pi} \int \hat{G}(\mathbf{x}_r, \mathbf{x}_s, \omega) \hat{F}(\omega) e^{i\omega t} d\omega + \\ & \frac{1}{2\pi} \int \hat{G}(\mathbf{x}_r, \mathbf{y}_*, \omega) \hat{G}(\mathbf{y}_*, \mathbf{x}_s, \omega) \rho \hat{F}(\omega) e^{i\omega t} d\omega \end{aligned}$$

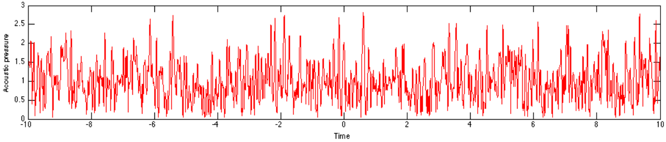
The Born Approximation
(this is a linear approximation!)

Imaging With Noise

- This is well understood when we know exactly the signal we send.



- What about if we send a random signal like white noise?



Cross-correlations

My work revolves around cross-correlations of noisy signals.

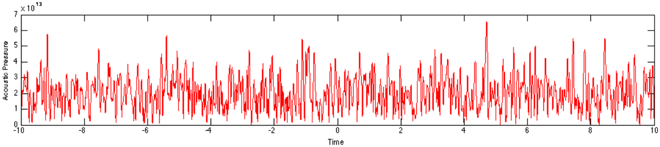
Using a noisy (literally noisy) signal F which is *a stochastic process* (means random signal) we record a random signal (*stochastic process*) $u(\mathbf{x}_r, t)$. Compute cross-correlations (averages):

$$M(\tau) = \frac{1}{T}\int_0^T \bar{F}(t)u(\mathbf{x}_r, t+\tau)dt.$$

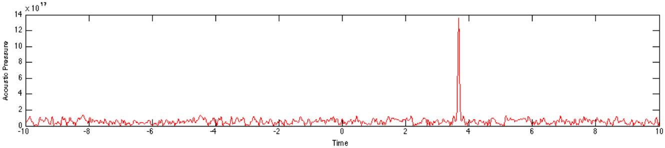
This is the average value of the single variable function $\bar{F}(t)u(\mathbf{x}_r, t+\tau)$ over the interval $[0, T]$.

Migration

Received signal $u(\mathbf{x}_r, t)$:



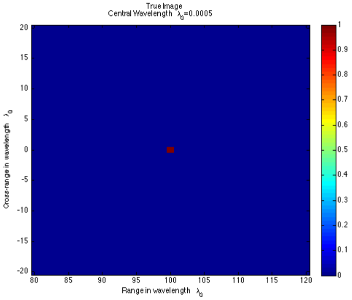
Cross-correlated signal $M(\tau)$:



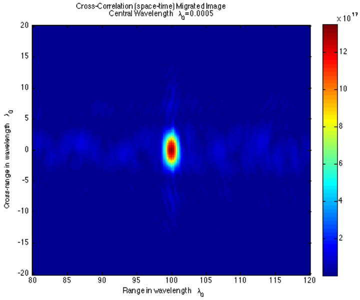
Evaluating $M(\tau)$ at the correct times we can find where a reflector may be located.

Migration

True reflector location



Reconstructed image



Imaging With Noise

Resolution Analysis: We study imaging a "point" reflector because it lets us determine how much we can trust imaging techniques in the real world (i.e. ultrasound). Useful in noninvasive medical imaging, and nondestructive testing of materials.

Tools from Math 1320:

- Partial derivatives
- Linear approximations
- Multivariable functions
- Average value of functions

“When am I ever going to use this?”

Goal: Encourage friends, family, and children that math is extremely useful and necessary to solve real world problems.

Thank you!

Thank you for a great semester- good luck with all your finals and of course...

HAGS!

[1] P. Bardsley and F. Guevara Vasquez. Restarted inverse Born series for the Schrödinger problem with discrete internal measurements. Inverse Problems, 30(4):045014, 2014. doi: 10.1088/0266-5611/30/4/045014. URL http://stacks.iop.org/0266-5611/30/i=4/a=045014.

[2] K. Barmak, E. Eggeling, M. Emelianenko, Y. Epshteyn, D. Kinderlehrer, R. Sharp, and S. Ta'asan. An entropy based theory of the grain boundary character distribution. Discrete Contin. Dyn. Syst., 30(2):427–454, 2011. ISSN 1078-0947. doi: 10.3934/dcds.2011.30.427. URL http://dx.doi.org/10.3934/dcds.2011.30.427.

[3] L. Borcea, G. Papanicolaou, and C. Tsogka. Theory and applications of time reversal and interferometric imaging. Inverse Problems, 19(6):S139–S164, 2003. ISSN 0266-5611. doi: 10.1088/0266-5611/19/6/058. URL http://dx.doi.org/10.1088/0266-5611/19/6/058. Special section on imaging.