Nerdy is the new black

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January 20, 2014

Overview

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  - Weather forecasting

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  - Spectrum analysis of acoustic signals
- The Fundamental Theorem of Calculus

Weather forecasting

The temperature $T(t)$ in a given area (SLC) can be described as

$$\frac{dT}{dt} = f(t, T, \text{wind}, \text{cloud cover}, \text{humidity}, \ldots)$$

where f is some function of all of those variables. As a *simple* example consider

$$\frac{dT}{dt} = 6\sin\left(\frac{1}{5}t^{\frac{11}{10}}\right) - 0.001t, \qquad T(0) = 40$$

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Problem: We can't integrate this! But... we could use an approximation of this function, which we find from a linearization.

Linearized differential equation

Using the linearization formula

$$f(t) \approx f(a) + f'(a)(t-a) = L(t)$$

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$$\frac{dT}{dt} = f(t) \approx L(t)$$

and we can solve the approximate differential equation to find

$$T(t) = \int f(t)dt + C \approx \int L(t)dt + C.$$

The linearization near time equal to zero of

$$f(t) = 6\sin\left(\frac{1}{5}t^{\frac{11}{10}}\right) - 0.001t.$$

is given by

$$L(t) = f(0) + f'(0)(t) = -0.001t.$$

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and using the fact that $T(0) = 40$ we find a temperature model

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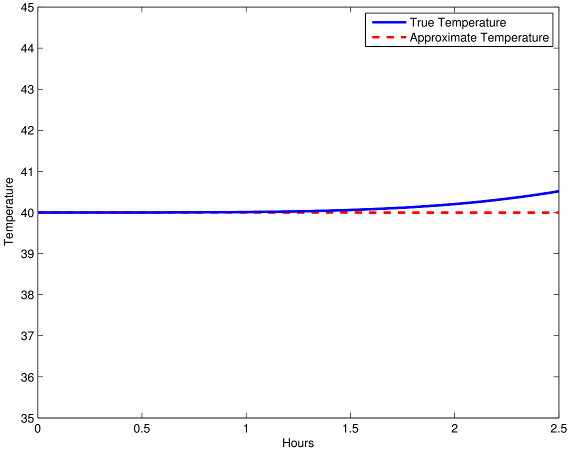
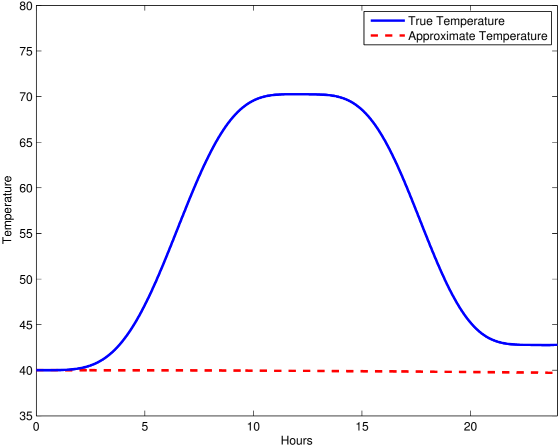


Why can't we trust the 7-day forecast?

This linearized approximation is great for short times (within 2 hours in this case).



Why can't we trust the 7-day forecast?

The linearized approximation breaks down as we move further away from time zero. So in general forecasting is hard for long periods of time because of the approximations to the model.

Optimal design of a climbing cam

The curved shape of a climbing cam

Optimal design of a climbing cam

The curved shape of a climbing cam is described by a sector of what is known as a logarithmic spiral $r = e^{\mu\theta}$ where r is the radius on the spiral, θ is the angle in radians, and μ describes how fast the spiral opens up.

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If this camming unit is placed inside a parallel crack we can look at a force diagram to understand what keeps climbers alive. Here the contact angle is what is most important for our purposes since it helps describe how much force we can expect on the center pin of the cam for a given load T.

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$$r = e^{\mu\theta}$$

and we have $y = r\sin(a)$ and $x = r\cos(a)$.

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Now instead of thinking of the spiral curve in polar coordinates $r = e^{\mu\theta}$, we can think of the curve being described in Euclidean (standard) coordinates as a function $y(x)$. By the relationship of polar coordinates we have

$$r^2 = e^{2\mu\theta}$$

$$x^2 + y^2 = e^{2\mu \tan^{-1}(y/x)}.$$

Implicit differentiation gives

$$2x + 2yy' = e^{2\mu \tan^{-1}(y/x)} \frac{2\mu}{x^2 + y^2}(xy' - y)$$

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Set the bottom to zero to find a vertical tangent line! Everything simplifies to

$$\mu = \tan(a) \iff a = \tan^{-1}(\mu).$$

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If we know μ, and we know the amount of force $|F|$ which will make the cam fail, we can find out how much of load the device can handle (i.e. how far we can fall before the unit fails).

Fourier analysis

What if we could build a complicated function $f(x)$ out of a bunch of other relatively simple functions? For example:

$$f(x) = a_0 + a_1 \cos(\pi x) + b_1 \sin(\pi x) + a_2 \cos(2\pi x) + b_2 \sin(2\pi x) + \cdots$$

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where

$$a_0 = \frac{1}{2} \int_0^2 x^2 dx = \frac{4}{3}$$
$$a_n = \int_0^2 x^2 \cos(n\pi x) dx = \frac{4}{n^2\pi^2}$$
$$b_n = \int_0^2 x^2 \sin(n\pi x) dx = \frac{-4}{n\pi}$$

So in a very general form we can write x^2 as

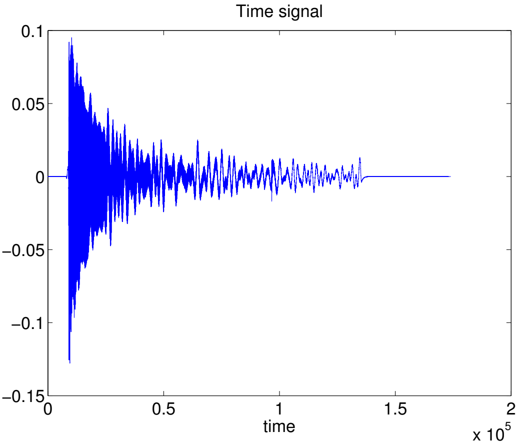
$$\begin{aligned} x^2 &= \sum_{n=0}^{\infty} \left(a_n \cos(n\pi x) + b_n \sin(n\pi x)\right) \\ &= \frac{4}{3} + \frac{4}{\pi^2}\cos(\pi x) + \frac{-4}{\pi}\sin(\pi x) + \\ &\quad \frac{1}{\pi^2}\cos(2\pi x) + \frac{-2}{\pi}\sin(2\pi x) + \cdots \end{aligned}$$

Fourier series

This function is easy already ($f(x) = x^2$), but this will be very useful for more difficult functions!

Spectrum analysis

A recording of a musical instrument is a time signal which may look like



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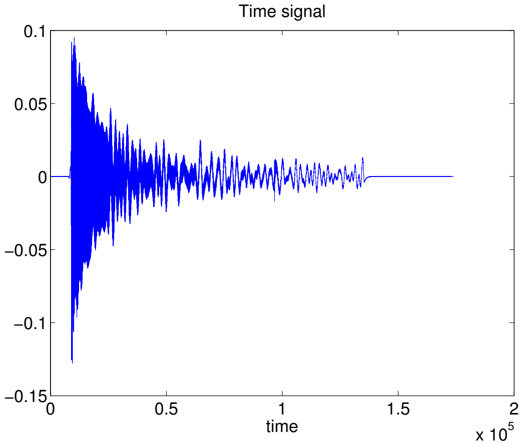

This is just a function $f(t)$ - Let's use Fourier series!

If T denotes the final time of the acoustic signal we are going to reconstruct the time function $f(t)$ as

$$f(t) = \sum_{n=0}^{\infty} a_n \cos(n\pi t) + b_n \sin(n\pi t),$$

where

$$a_0 = \frac{1}{T} \int_0^T f(t) dt$$

$$a_n = C_1 \int_0^T f(t) \cos(n\pi t) dt$$

$$b_n = C_2 \int_0^T f(t) \sin(n\pi t) dt$$

where we know exactly what C_1 and C_2 are.

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So we use a finite amount N^* of them as an approximation:

$$f(t) \approx \sum_{n=0}^{N^*} a_n \cos(n\pi t) + b_n \sin(n\pi t)$$

and we obtain the following sounds.

The Fundamental Theorem of Calculus

Recall if

$$d(t) = \int_0^t v(\tau) d\tau$$

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then

$$d'(t) = v(t).$$

If v represents velocity (mph), then $d(t)$ represents the distance traveled (mi) over the time interval $[0, t]$.

Let's say $v(t) = 80$ mph, then in 1 hour we have traveled

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Wow, if we are traveling 80mph, in one hour we will have traveled 80 miles! Who knew?!

Thanks for a great semester! Good luck on your finals and have a great winter break!