1. **Isothermals** The gas law for a fixed mass \( m \) of an ideal gas at absolute temperature \( T \), pressure \( P \), and volume \( V \) is \( PV = mRT \), where \( R \) is the gas constant.

(a) The level curves of \( T \) are called isothermals because at all points on such a curve the temperature is the same. Sketch some of the isothermals given by the ideal gas law for a gas constant \( R = 0.25 \) and a mass \( m = 1.25 \). **Hint:** The temperature can be viewed as a function of pressure and volume.

(b) Show that

\[
\frac{\partial P}{\partial V} \frac{\partial V}{\partial T} \frac{\partial T}{\partial P} = -1
\]

(c) Show that, for an ideal gas,

\[
T \frac{\partial P}{\partial T} \frac{\partial V}{\partial T} = mR
\]

**Solution:**

(a) The plot below is a contour plot of the Temperature \( T \) for varying values of volume \( V \) and pressure \( P \). The dark lines represent contours which are constant in \( T \), thus they represent the isothermals of the ideal gas law for the chosen parameters \( R = 0.25 \) and \( m = 1.25 \).

(b) \( P = \frac{mRT}{V} \) so \( \frac{\partial P}{\partial V} = -\frac{mRT}{V^2}, \ V = \frac{mRT}{P} \) so \( \frac{\partial V}{\partial T} = \frac{mR}{P}, \ T = \frac{PV}{mR} \) so \( \frac{\partial T}{\partial P} = \frac{V}{mR} \). Thus \( \frac{\partial P}{\partial V} \frac{\partial V}{\partial T} \frac{\partial T}{\partial P} = -1 \), since \( PV = mRT \). Notice this is...
exactly negative of the quantity we would arrive at if we treated all of the terms $\partial P$, $\partial V$, and $\partial T$ as fractions and canceled “like” terms. Thus we see derivatives are “almost” like fractions but still something a little different.

(c) By part (a), $PV = mRT \implies P = \frac{mRT}{V}$, so $\frac{\partial P}{\partial T} = \frac{mR}{V}$. Also, $PV = mRT \implies V = \frac{mRT}{P}$ and $\frac{\partial V}{\partial T} = \frac{mR}{P}$. Since $T = \frac{PV}{mR}$, we have $T\frac{\partial P}{\partial T} \frac{\partial V}{\partial T} = \frac{PV}{mR} \frac{mR}{P} = mR$. 
2. **Frost Penetration** In a study of frost penetration it was found that the temperature $T$ at time $t$ (measured in days) at a depth $x$ (measured in feet) can be modeled by the function

$$T(x, t) = T_0 + T_1 e^{-\lambda x} \sin(\omega t - \lambda x)$$

where $\omega = 2\pi/365$ and $\lambda$ is a positive constant.

(a) Find $\frac{\partial T}{\partial x}$. What is its physical significance?

(b) Find $\frac{\partial T}{\partial t}$. What is its physical significance?

(c) Show that $T$ satisfies the heat equation $T_t = kT_{xx}$ for a certain constant $k$.

(d) If $\lambda = 0.2$, $T_0 = 0$, and $T_1 = 10$, graph $T(x, t)$.

(e) What is the physical significance of the term $-\lambda x$ in the expression $\sin(\omega t - \lambda x)$?

**Solution:**

(a)

$$\frac{\partial T}{\partial x} = T_1 e^{-\lambda x} \left[ \cos(\omega t - \lambda x) (-\lambda) - \lambda e^{-\lambda x} \sin(\omega t - \lambda x) \right]$$

$$= -\lambda T_1 e^{-\lambda x} \left[ \sin(\omega t - \lambda x) + \cos(\omega t - \lambda x) \right]$$

This quantity represents the rate of change of temperature with respect to the depth below the surface at a given time $t$.

(b)

$$\frac{\partial T}{\partial t} = T_1 e^{-\lambda x} \left[ \cos(\omega t - \lambda x) (\omega) \right] = \omega T_1 e^{-\lambda x} \cos(\omega t - \lambda x)$$

This quantity represents the rate of change of temperature with respect to time at a fixed depth $x$.

(c)

$$T_{xx} = \frac{\partial}{\partial x} \left( \frac{\partial T}{\partial x} \right)$$

$$= -\lambda T_1 e^{-\lambda x} \left[ \cos(\omega t - \lambda x) (-\lambda) - \sin(\omega t - \lambda x) (-\lambda) \right] + \cdots$$

$$+ e^{-\lambda x} (-\lambda) \left[ \sin(\omega t - \lambda x) + \cos(\omega t - \lambda x) \right]$$

$$= 2\lambda^2 T_1 e^{-\lambda x} \cos(\omega t - \lambda x)$$

From part (b) we saw, $T_t = \omega T_1 e^{-\lambda x} \cos(\omega t - \lambda x)$. So with $k = \frac{\omega}{2\lambda^2}$, the function $T$ satisfies the heat equation.
(d) The term $-\lambda x$ is a phase shift: it represents the fact that since heat diffuses slowly through soil, it takes time for changes in the surface temperature to affect the temperature at deeper points. As $x$ increases, the phase shift also increases. For example, when $\lambda = 0.2$, the highest temperature at the surface is reached when $t \approx 91$, whereas at a depth of 5 feet, the peak temperature is attained at $t \approx 149$, and at a depth of 10 feet, at $t \approx 207$. 