HW9 **NAME: ....** **due 3/28/2018**

1. Solve Laplace's equation $\phi_{xx} + \phi_{yy} = 0$ in the domain D

   (a) $D = \{z : |z-1| > 1, \ |z-2| < 2\}$;
   boundary conditions: $\phi(x, y) = a$ when $|z-1| = 1$ and $\phi(x, y) = b$ when $|z-2| = 2$.
   Suggestion: Conformally transform the domain between the circles onto a domain between two parallel straight lines and use new obvious symmetry to reduce the PDE to an ODE.

   (b) $D = \{z : |z| < 1, \ |z-i| < \sqrt{2}\}$;
   boundary conditions: $\phi(x, y) = a$ when $|z| = 1$ and $\phi(x, y) = b$ when $|z-i| = \sqrt{2}$.
   Suggestion: Explore the maps $w = \frac{z-1}{z+1}$ and $\zeta = \ln w$.

2. Let $F(\omega)$ be the Fourier transform of $f(x)$; $-\infty < x < \infty$, $-\infty < \omega < \infty$.
The smoother $f(x)$ is, the faster $F(\omega)$ vanishes at ∞.
The faster $f(x)$ vanishes at ∞, the smoother $F(\omega)$ is.

(a) Prove the Riemann-Lebesgue lemma: If a function $f(x)$ is of the class $L^1(-\infty, \infty)$, then its Fourier transform is continuous and approaches zero as $\omega \to \pm\infty$.

(b) Show that if $f(x)$ and its derivatives up to the order n are of the class $L^1(-\infty, \infty)$, then $F(\omega) = o(\omega^{-n})$, $\omega \to \infty$.

(c) Show that if the functions $f(x)$, $xf(x), \ldots, x^m f(x)$ are of the class $L^1(-\infty, \infty)$, then $F(\omega)$ has continuous derivatives up to the order m, and each of them vanishes at infinity.

Hint: Differentiation in the x-space means multiplication in the Fourier space.

(d) Let

$$f(x) = \begin{cases} 1, & |x| < \pi, \\ 0, & \text{otherwise.} \end{cases}$$

How fast does $F(\omega)$ vanishes at ∞?

(e) Let

$$f(x) = \begin{cases} \sin x, & |x| < \pi, \\ 0, & \text{otherwise.} \end{cases}$$

How fast does $F(\omega)$ vanishes at ∞?

Suggestion: (d) and (e) are particular cases of (b), but (b) gives only an estimate; you probably need to find the Fourier transforms explicitly.

3. Consider the boundary value problem for the heat equation on the whole line $-\infty < x < \infty$:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = f(x), \qquad u \to 0 \text{ as } x \to \pm\infty$$

(parameter $k > 0$ is the thermal diffusivity, $f(x)$ is a given function, $f(x) \to 0$ as $x \to \pm\infty$).

(a) Derive the formula for its solution as a convolution of Green's function with the initial condition $f(x)$.

(b) Show that the heat equation implies infinite propagation speed.

(c) Show continuous dependence on the initial condition: A small variation in the initial condition causes small variation in the solution.

(d) Why can there be no (meaningful) continuous dependence on the initial condition for the heat equation with the wrong sign ($u_t = -k\,u_{xx},\ k > 0$)?

4. **How the Fourier transform leads to analytic functions.** Consider a continuous function $f(x)$, $-\infty < x < \infty$. Suppose

$$|f(x)| < Ae^{ax} \quad \text{if} \quad 0 < x < \infty, \qquad |f(x)| < Be^{bx} \quad \text{if} \quad -\infty < x < 0$$

(A, a, B, b are some real constants; A, B are positive; a, b can have any signs, but $a < b$.)

(a) Show that the Fourier transform

$$F(\mu) = \int_{-\infty}^{\infty} f(x)\, e^{i\mu x}\, dx$$

is an analytic function in some strip. Find this strip.

Suggestion: Check that the function $f(x)e^{-\sigma x}$ exponentially decays at $x \to \pm\infty$ for certain σ.

(b) Show that the original can be restored by the complex inverse Fourier transform

$$f(x) = \int_{i\sigma-\infty}^{i\sigma+\infty} F(\mu)\, e^{-i\mu x}\, \frac{d\mu}{2\pi}$$

(σ is real; the integration here is along an arbitrary line $\operatorname{Im}\mu = \sigma$ that belongs to that analyticity strip).

Suggestion: Consider the usual real Fourier transform of the function $f(x)e^{-\sigma x}$.

(c) Assuming $a < b$, show that the Fourier transform

$$\text{of} \qquad f(x) = \begin{cases} e^{ax}, & x > 0 \\ e^{bx}, & x < 0 \end{cases} \qquad \text{is} \qquad F(\mu) = -\frac{1}{a + i\mu} + \frac{1}{b + i\mu}\,.$$

In what domain of the complex μ-plane does this formula hold?

(d) The function $F(\mu)$ in (c) is analytic in the complex plane w/o only two points $\mu = ia$ and $\mu = ib$. What is the inverse transform $\int_{i\sigma-\infty}^{i\sigma+\infty} F(\mu)\, e^{-i\mu x}\, \frac{d\mu}{2\pi}$ for different values of σ (other than a and b)?