HW8 NAME: .... due Wednesday, 3/15/2018

1. Let an analytic function $w = F(z)$ map a domain D one-to-one onto domain Δ. Show that $F'(z_0) \neq 0$ for any $z_0 \in D$.

2. Show that

   (a) a bilinear map takes circles into circles.
   Suggestion:

      i. Any bilinear map is a composition of linear maps and inversion.
      ii. A linear map takes circles into circles.
      iii. The inversion $w = 1/z$ takes circles into circles. [This is trivial for a circle centered at the origin. But it is true for any circle.]

   (b) any circle in the z-plane can be mapped onto any circle in the ζ-plane by a suitable bilinear map.
   Suggestion: Note that any circle is determined by three points and consider the equation

$$\frac{(\zeta - \zeta_1)(\zeta_3 - \zeta_2)}{(\zeta_1 - \zeta_2)(\zeta - \zeta_3)} = \frac{(z - z_1)(z_3 - z_2)}{(z_1 - z_2)(z - z_3)}.$$

   See that it defines a bilinear transformation $w(z)$, which takes points z_1, z_2, z_3 to points w_1, w_2, w_3. Now use part (a) to check that the entire circle passing through z_1, z_2, z_3 is mapped onto the circle passing through w_1, w_2, w_3.

3. Prove **the Schwartz Lemma:** Let $f(z)$ be analytic in the unit disc $U : |z| < 1$.
If $|f(z)| \leq 1$ in U and $f(0) = 0$, then $|f(z)| \leq |z|$ in U.

Suggestion: Consider function $h(z) = \frac{f(z)}{z}$. Show that it is analytic in U (the singularity at $z = 0$ can be removed).
Take an arbitrary $\rho \in (0, 1)$. Apply the maximum principle to $h(z)$ to show that $|h(z)| \leq 1/\rho$ when z is in the disc $U_\rho : |z| < \rho$.
Finally, take the limit as $\rho \to 1$.

4. Show that a conformal map of a disc onto a disc is necessarily bilinear.

Suggestion: Consider an arbitrary disc in the z-plane and another arbitrary disc in the ζ-plane. Riemann says that there is a conformal transformation $\zeta = f(z)$ of the first disc onto the second one. You need to show that this conformal map is necessarily bilinear.

(a) Some point z_0 of the first disc is taken to some point ζ_0 of the second disc.
There is a bilinear transformation $z \to \tilde{z}$ of the z-plane disc onto the unit disc $|\tilde{z}| < 1$; herewith z_0 is taken to the origin $\tilde{z}_0 = 0$.
Similar, there is a bilinear transformation $\zeta \to \tilde{\zeta}$ of the ζ-plane disc onto the unit disc $|\tilde{\zeta}| < 1$; ζ_0 is taken to the origin $\tilde{\zeta}_0 = 0$.

(b) Schwartz lemma shows that if a conformal transformation $\zeta = f(z)$ maps the unit disc onto itself,
so that the origin of z-plane is taken to the origin of the ζ-plane,
then the transformation is just a rotation about the origin: $f(z) = e^{i\beta} z$ (β is a real number, independent of z).
[Indeed, according to the Schwartz lemma, $|\zeta| \le |z|$.
Applying the Schwartz lemma to the inverse transformation $z = f^{-1}(\zeta)$, we find $|z| \le |\zeta|$.
Thus, $\left|\frac{f(z)}{z}\right| = 1$, and so, $\frac{f(z)}{z}$ =const.]

5. Solve Laplace's equation

$$\phi_{xx} + \phi_{yy} = 0 \quad \text{in the domain between two circles} \quad x^2 + y^2 = 1 \quad \& \quad (x-1)^2 + y^2 = \left(\frac{5}{2}\right)^2$$

subject to the boundary condition

$$\phi(x,y) = a \text{ when } x^2 + y^2 = 1 \quad \text{and} \quad \phi(x,y) = b \text{ when } (x-1)^2 + y^2 = \left(\frac{5}{2}\right)^2$$

(a and b are real parameters).

Suggestion: Conformally transform the domain between non-concentric circles onto a domain between concentric circles and use new symmetry to reduce the PDE to an ODE.