HW7 **NAME: ....** **due 3/7/2018**

1. *(1a) is a particular case of the general situation (1b)*

   (a) The function $f(z)$ is analytic in the entire complex plane except at $z = i/2$, where it has a simple pole, and at $z = 2$, where it has a double pole. It is known that

$$\begin{aligned} \oint_{|z|=1} f(z)dz &= 2\pi i\,, \\ \oint_{|z|=3} f(z)dz &= 0\,, \\ \oint_{|z|=3} f(z)(z-2)dz &= 0\,, \end{aligned}$$

   and $f(z)$ is bounded at infinity (i.e. $\exists M > 0,\ \exists R > 0 : |z| > R \Rightarrow |f(z)| < M$).

   Find $f(z)$ (unique up to an arbitrary additive constant).

   Suggestion: Find an examle $f_0(z)$ of such function and use the Liouville theorem to show that $f(z) - f_0(z)$ is constant.

(b) Prove that if all singularities of an analytic function in the extended complex plane are poles, then this function is a rational function (i.e. the ratio of two polynomials).

Suggestion: First show that the function can have only finite number of poles. Then for each pole at finite z take the negative power part of the corresponding Laurent expansion; for $z = \infty$, take the positive power part; add all of them to get a rational function. Finally, apply the Liouville theorem.

2. *Argument principle to locate zeros and poles of an analytic function.* Prove:

(a) If $f(z)$ has a zero at z_0 of order m, then

$$\text{Res}\left[\frac{f'(z)}{f(z)}\ ;\ z = z_0\right] \;=\; m$$

(b) If $f(z)$ has a pole at z_0 of order n, then

$$\text{Res}\left[\frac{f'(z)}{f(z)}\ ;\ z = z_0\right] \;=\; -n$$

(c) Suppose $f(z)$ is analytic inside and on a simple closed curve C, except for finite number of poles inside C, and moreover, $f(z) \neq 0$ on C.
Then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)} dz \;=\; N_0 - N_\infty\ ,$$

where N_0 and N_∞ are (respectively) the number of zeros and the number of poles of $f(z)$ inside C. Both zeros and poles are to be counted with their orders (multiplicities).

(d) **Argument Principle.**

$$N_0 - N_\infty \;=\; \frac{1}{2\pi}\Delta_C \arg\{f(z)\} \;=\; \left\{\begin{array}{l}\text{the number of times}\\ \text{the point } w = f(z) \text{ surrounds the origin } w = 0\\ \text{as } z \text{ describes } C\end{array}\right\}$$

(the conditions of the previous statement are to be satisfied,
$\Delta_C \arg\{f(z)\}$ denotes the total variation of $\arg\{f(z)\}$ as the point z describes the curve C).

3. *Applications of the Argument Principle*

(a) Use the argument principle to determine the number of solutions of the equation $z^5 + 1 = 0$ in the first quadrant (with positive real and imaginary parts of z.)

(b) Prove the **fundamental theorem of algebra**, using the argument principle.

Suggestion: Apply the argument principle for a very big circle, so that all lower powers are well dominated by the highest power.

(c) Prove **Rouche's theorem**: If the functions $f(z)$ and $\phi(z)$ are analytic inside and on simple closed curve C and if the strict inequality $|\phi(z)| < |f(z)|$ holds for all $z \in C$, then the function $f(z) + \phi(z)$ has exactly as many zeros (counting their multiplicities) inside C as the function $f(z)$.

Suggestion: First, show that the functions $f(z)$ and $f(z) + \phi(z)$ do not vanish on C.
Then show that $\Delta_C \arg\{f(z)\} = \Delta_C \arg\{f(z) + \phi(z)\}$

(d) Determine the number of roots (counting their multiplicities) of the equation $z^4 + 3z^3 + 6 = 0$ inside the circle $|z| = 2$.

(e) Determine the number of roots (counting their multiplicities) of the equation $2z^5 - 6z^2 + z + 1 = 0$ inside the annulus $1 \le |z| \le 2$.

(f) Determine the number of roots of the equation $z^4 + 3z^3 + 6 = 0$ inside the circle $|z| = 2$.
Does this equation have roots of multiplicity $m > 1$?