HW5 **NAME: ....** **due Wednesday, 2/14/2018**

1. Let z_0 be an isolated singularity of an analytic function $f(z)$. Near z_0, the function $f(z)$ can be represented by its Laurent series (L)

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

Prove:

(a) If (L) has no negative powers, then z_0 is a removable singularity.

(b) If z_0 is a removable singularity, then (L) has no negative powers.
Suggestion: You should actually prove a little stronger statement:
If $f(z)$ is bounded in some neighborhood around z_0, then (L) has no negative powers.
Idea of the proof: Consider the formula for the (L) coefficients given as the integral over a small circle $|z - z_0| = R \to 0$; estimate for $n < 0$. This is similar to the proof of the Liouville theorem.

(c) If (L) has a finite number of negative powers, then z_0 is a pole.

(d) If z_0 is a pole, then (L) has a finite number of negative powers.
Suggestion: Expand $g(z) = 1/f(z)$ in the Taylor series around z_0.

(e) If (L) has infinitely many negative powers, then z_0 is an essential singularity.

(f) If z_0 is an essential singularity, then (L) has infinitely many negative powers.
Suggestion: The last two statements follow from the previous ones.

2. Prove Kasorati-Sokhotsky-Weierstrass Theorem:

   If an analytic function $f(z)$ has an essential singularity at z_0,
   then $f(z)$ comes arbitrarily close to *any* complex value in every neighbourhood of z_0;
   in other words,
   $\forall\ \epsilon > 0,\ \ \forall \delta > 0$, and $\forall$ complex number w, $\exists$ a complex number z such that
   $|z - z_0| < \delta$ and $|f(z) - w| < \epsilon$.

   Suggestion: Assume the contrary and consider function $g(z) = [f(z) - w]^{-1}$.

3. Show that

$$\int_0^{2\pi} \frac{d\phi}{(p + q\cos\phi)^2} = \frac{2\pi p}{(p^2 - q^2)^{3/2}} \qquad \text{(with parameters } p > q > 0\text{).}$$

4. Show that

$$\int_0^{\infty} \frac{x^m}{x^n+1}\,dx = \frac{\pi}{n\,\sin\frac{\pi(m+1)}{n}} \qquad \text{with integer parameters } m \text{ and } n, \quad n-2 \geq m \geq 0.$$

Suggestion: Notice that the expression under the integral sign is multiplied by a constant factor when x is changed to $xe^{i2\pi/n}$. Consider residue integration in the sector between $z = r$ and $z = re^{i2\pi/n}$ $(0 \leq r < \infty)$. The integrand has only one pole in this sector.