1. Let $z_0$ be an isolated singularity of an analytic function $f(z)$. Near $z_0$, the function $f(z)$ can be represented by its Laurent series (L)

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$ 

Prove:

(a) If (L) has no negative powers, then $z_0$ is a removable singularity.

(b) If $z_0$ is a removable singularity, then (L) has no negative powers.  
    \textit{Suggestion:} You should actually prove a little stronger statement:  
    If $f(z)$ is bounded in some neighborhood around $z_0$, then (L) has no negative powers.  
    \textit{Idea of the proof:} Consider the formula for the (L) coefficients given as the integral over a small circle $|z - z_0| = R \to 0$; estimate for $n < 0$. This is similar to the proof of the Liouville theorem.

(c) If (L) has a finite number of negative powers, then $z_0$ is a pole.

(d) If $z_0$ is a pole, then (L) has a finite number of negative powers.  
    \textit{Suggestion:} Expand $g(z) = 1/f(z)$ in the Taylor series around $z_0$.

(e) If (L) has infinitely many negative powers, then $z_0$ is an essential singularity.

(f) If $z_0$ is an essential singularity, then (L) has infinitely many negative powers.  
    \textit{Suggestion:} The last two statements follow from the previous ones.
2. Prove Kasorati-Sokhotsky-Weierstrass Theorem:

If an analytic function $f(z)$ has an essential singularity at $z_0$,
then $f(z)$ comes arbitrarily close to any complex value in every neighbourhood of $z_0$;
in other words,
$\forall \epsilon > 0, \forall \delta > 0$, and $\forall$ complex number $w$, $\exists$ a complex number $z$ such that
$|z - z_0| < \delta$ and $|f(z) - w| < \epsilon$.

_Suggestion:_ Assume on the contrary and consider bounded function $g(z) = [f(z) - w]^{-1}$; expand it in the Taylor series.
Express $f(z)$ via $g(z)$ and arrive at contradiction that $f(z)$ either has a pole or removable singularity at $z_0$. 
3. The (single-valued) function \( f(z) = \sqrt{z(z - 1)(z - 4)} \) is defined by

branch cuts \( z = -it, \quad z = 1 + it, \quad z = 4 - it \quad (0 \leq t < \infty) \)

and value \( f(2) = -2i \).

Evaluate

(a) \( f(-3) \),
(b) \( f(1/2) \),
(c) \( f(5) \).
4. The (single-valued) function

\[ f(z) = \frac{(z - 16)^{1/3}(z - 20)^{1/2}}{z - 1} \]

is defined by

\[
\text{branch cuts } z = 16 + it, \quad z = 20 - it \quad (0 \leq t < \infty)
\]

and value \( f(24) = \frac{4}{23} \).

Evaluate \( \oint_C f(z)\,dz \) over the circle \( C : |z - 2| = 2 \).
5. We are given multivalued analytic function

\[ f(z) = [(z - 1)(z - 2)]^{1/3}. \]

What branch cuts would you make to separate single-valued branches? Would the cut from \( z = 1 \) to \( z = 2 \) along the real axis work?
6. The (single-valued) function \( f(z) = \sqrt{z(z-1)} \) is defined by the branch cut \( z = t \ (0 \leq t \leq 1) \) and the condition that \( f \) is positive for real \( z > 1 \).

   (a) Find the first three non-zero terms of its Laurent series (centered at \( z_0 = 0 \)) in the domain \( |z| > 1 \).
   (b) Where does the series converge?
7. Evaluate the integral
\[ \int_C z^{-1/3} \, dz \]
over the path \( C \) traversing from \( z = 1 + i \) to \( z = 1 - i \) along the following composite curve \( C = C_1 + C_2 + C_3 \)

- \( C_1 \): hyperbola \( 2y^2 - x^2 = 1 \) from \( z = 1 + i \) to \( z = -2 + i \sqrt{5/2} \),
- \( C_2 \): segment \( x = -2 \) from \( z = -2 + i \sqrt{5/2} \) to \( z = -2 - i \sqrt{5/2} \),
- \( C_3 \): hyperbola \( 2y^2 - x^2 = 1 \) from \( z = -2 - i \sqrt{5/2} \) to \( z = 1 - i \).

Here \( z^{-1/3} \) denotes the branch defined by the branch cut along the positive \( x \)-axis and the condition that it is real when \( z \) is negative.
8. Show that

\[
\int_0^{2\pi} \frac{d\phi}{(p + q \cos \phi)^2} = \frac{2\pi p}{(p^2 - q^2)^{3/2}}
\]

(with constant parameters \( p > q > 0 \)).
9. Show that

\[ \int_{0}^{\infty} \frac{x^m}{x^n + 1} \, dx = \frac{\pi}{n \sin \frac{\pi(m+1)}{n}} \]

with integer parameters \( m \) and \( n \), \( n - 2 \geq m \geq 0 \).

**Suggestion:** Notice that the expression under the integral sign is multiplied by a constant factor when \( x \) is changed to \( xe^{\frac{2\pi}{n}} \). Consider residue integration in the sector between \( z = r \) and \( z = re^{\frac{2\pi}{n}} \) (\( 0 \leq r < \infty \)). The integrand has only one pole in this sector.
10. **Jordan’s lemma** to neglect integrals

\[ I(R) = \int_{C_R} f(z) e^{iaz} \, dz \quad (a > 0) \]

over “large” semi-circle \( C_R \) : \( z = R e^{i\phi} \) \( (0 \leq \phi \leq \pi) \), \( R \to \infty \).

**Prove:** If on \( C_R \), \( |f(z)| < F(R) \to 0 \) [for some function \( F(R) \)], then \( I(R) \to 0 \).