1. (a) Prove the **Mean Value Theorem:**

The value of analytic function $f(z_0)$ equals the arithmetic mean average of the values of this function on a circle with center at $z_0$

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta})d\theta$$

(the circle should lie in the analyticity domain, but otherwise the radius $R$ is arbitrary).

(b) Prove: If $f(z)$ is analytic in a domain $D$, then $|f(z)|$ cannot attain a strict local maximum in $D$. 
2. (a) Prove:
   If \( f(z) \) is analytic in a domain \( D \), and \( |f(z)| \) is constant in \( D \), then \( f(z) \) is constant in \( D \).

   Idea: If \( |f(z)| \equiv M \neq 0 \), consider function \( \phi = \ln f \) and apply Cauchy-Riemann conditions.

(b) Prove the **Maximum Modulus Principle**:
Suppose \( f(z) \) is analytic in a domain \( D \).
The only way \( |f(z)| \) can attain maximum in \( D \) is when \( f(z) \equiv \text{const} \) in \( D \).

   Suggestion: Assume on the contrary \( |f(z)| \) attains maximum \( M \) at some point \( z_0 \in D \), and \( f(z) \) is not a constant.

   \( f(z) \) is analytic in some disk \( S = \{ z \in D : |z - z_0| < R_0 \} \) that belongs to \( D \). \( \exists z_1 \in S \), such that \( |f(z_1)| < M \). (Otherwise, \( |f(z)| = M \) in \( S \), and \( f(z) \equiv \text{const} \) in \( S \); then by the uniqueness Th, \( f(z) \equiv \text{const} \) in \( D \).) Now, use the Mean Value Th (from \#1) for the circle of radius \( R = |z_1 - z_0| \) and center at \( z_0 \). Some piece of this circle has \( |f(z)| < M \) (by continuity). So, you arrive at contradiction \( M < M \).

(c) Prove:
Let a function \( f(z) \) be analytic in a bounded domain \( D \), continuous in \( \bar{D} \) and not a constant.
Then \( |f(z)| \) attains its maximum on the boundary \( \partial D \) and nowhere inside.
3. (a) Give an example of a non-constant function \( f(z) \) analytic in a domain \( D \), so that \( |f(z)| \) attains a strict local minimum in \( D \).

(b) Prove: If \( f(z) \) is analytic in a domain \( D \), and \( f(z) \neq 0 \) for any \( z \in D \), then \( |f(z)| \) can attain minimum in \( D \) only if \( f(z) \) is a constant.
4. (a) Let $f(z)$ be analytic in a domain $D$.
   Definition: A point $z_0 \in D$ is called a zero of $f(z)$, if $f(z_0) = 0$.
   Prove that any analytic function (not identically equal to zero) can have only isolated zeros.
   (If $f(z_0) = 0$, then there exists a positive $\epsilon$ such that $f(z) \neq 0$ when $0 < |z - z_0| < \epsilon$.)
   
   *Suggestion:* See the proof of the Uniqueness Theorem.

(b) Can an analytic function have a non-isolated singularity?
5. Give an example of a power series (over non-negative integer powers) whose radius of convergence $R$ is

(a) finite $R \neq 0$,
(b) $R = \infty$,
(c) $R = 0$. 
6. (a) Show that the function

\[ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt \]

satisfies the functional equation \( \Gamma(z + 1) = z \Gamma(z) \) for any complex \( z \) whose real part is positive.

(b) Show that this function generalizes factorial to complex numbers, namely \( \Gamma(n) = (n-1)! \) for any positive integer \( n \).

(c) Does there exist another function \( F(z) \) which is also analytic in the right half-plane \( \text{Re}(z) > 0 \) and coincides with \( \Gamma(z) \) when \( z \) is any positive integer?
7. A function \( f(x) \) has two power representations in a neighborhood of \( x = 0 \)

\[
f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad f(x) = \sum_{n=0}^{\infty} b_n x^n
\]

Is it true that \( a_n = b_n \) for all \( n = 0, 1, 2, \ldots \)?

[If this is true, then we can find Taylor series in any way (in particular, without differentiation).]
8. Let the **Euler numbers** \( E_n \) be defined by the power series

\[
\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n.
\]

(a) What is the radius of convergence of this series?

(b) Determine the first six Euler numbers.

*Suggestion:* Do not differentiate. Instead, expand \( e^z \), \( \cosh z \), \( 1/\cosh z \) in Taylor series with center at the origin.
9. Suppose a complex function \( f(z) \) is differentiable in a domain \( D \) of the complex plane. Prove that if the domain \( D \) contains annulus

\[
A : \quad R_1 < |z - z_0| < R_2
\]

\((z_0 \text{ is an arbitrary complex point, which may or may not belong to } D)\) then the function \( f(z) \) can be represented by its Laurent series:

\[
f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad \text{where} \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta,
\]

and \( C \) is any simple closed curve in \( A \) enclosing \( z_0 \).

While making the proof, answer these questions:

(a) Where does this series converge to \( f(z) \)?

(b) Where does the series converge absolutely?

(c) Where does the series converge uniformly?

(d) Is it true that \( a_n n! = f^{(n)}(z_0) \) for positive \( n \)?

*Suggestion:* Fix \( z \) and consider a smaller annulus \( R'_1 < |z - z_0| < R'_2 \) that still contains \( z \). Using Cauchy’s formula, represent \( f(z) \) as the sum of two integrals: one over circle \( |z - z_0| = R'_1 \), another over circle \( |z - z_0| = R'_2 \). Then use geometric series expansions.

To have a little shorter writing, you can take \( z_0 = 0 \).