HW4 **NAME: ....** **due Wednesday, 2/7/2018**

1. **{*}**

   (a) Prove the **Mean Value Theorem:**
   The value of analytic function $f(z_0)$ equals the arithmetic mean average of the values of this function on a circle with center at z_0

   $$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) d\theta$$

   (the circle should lie in the analyticity domain, but otherwise the radius R is arbitrary).

   (b) Prove: If $f(z)$ is analytic in a domain D, then $|f(z)|$ cannot attain a strict local *maximum* in D.

   (c) Give an example of a non-constant function $f(z)$ analytic in a domain D, so that $|f(z)|$ attains a strict local *minimum* in D.

2. {*}

(a) Let $f(z)$ be analytic in a domain D. Definition: A point $z_0 \in D$ is called a zero of $f(z)$, if $f(z_0) = 0$. Prove that any analytic function (not identically equal to zero) can have only isolated zeros (if $f(z_0) = 0$, then there exists a positive ϵ such that $f(z) \neq 0$ when $0 < |z - z_0| < \epsilon$).

Suggestion: See the proof of the Uniqueness Theorem.

(b) Can an analytic function have a non-isolated singularity?

3. (a) Show that the function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

satisfies the functional equation $\Gamma(z+1) = z\Gamma(z)$ for any complex z whose real part is positive.

(b) Show that this function generalizes factorial to complex numbers,
namely $\Gamma(n) = (n-1)!$ for any positive integer n.

4. **{*}** Does there exist another function $F(z)$ which is also analytic in the right half-plane $\mathrm{Re}(z) > 0$ and coincides with $\Gamma(z)$ when z is any positive integer?

5. Give an example of a power series (over non-negative integer powers) whose radius of convergence R is

   (a) finite $R \neq 0$,

   (b) $R = \infty$,

   (c) $R = 0$.

6. A function $f(x)$ has two power representations in a neighborhood of $x = 0$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \text{and} \qquad f(x) = \sum_{n=0}^{\infty} b_n x^n$$

   Is it true that $a_n = b_n$ for all $n = 0, 1, 2, \ldots$?

   [If this is true, then we can find Taylor series in any way (in particular, without differentiation).]

7. **{*}** Let the **Euler numbers** E_n be defined by the power series

$$\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n.$$

   (a) What is the radius of convergence of this series?

   (b) Determine the first six Euler numbers.

   Suggestion: Do not differentiate. Instead, expand e^z, $\cosh z$, $1/\cosh z$ in Taylor series with center at the origin.

8. Suppose a complex function $f(z)$ is differentiable in a domain D of the complex plane. Prove that if the domain D contains annulus

$$A: \quad R_1 < |z - z_0| < R_2$$

(z_0 is an arbitrary complex point, which may or may not belong to D) then the function $f(z)$ can be represented by its Laurent series:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad \text{where} \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta,$$

and C is any simple closed curve in A enclosing z_0.

While making the proof, answer these questions:

(a) Where does this series converge to $f(z)$?

(b) Where does the series converge absolutely?

(c) Where does the series converge uniformly?

(d) Is it true that $a_n n! = f^{(n)}(z_0)$ for positive n?

(e) Where & why in the proof do you need to consider a smaller annulus $A' : \; R_1' < |z - z_0| < R_2'$ ($R_1' > R_1, \; R_2' < R_2$) compared to the original annulus $A: \; R_1 < |z - z_0| < R_2$?

Suggestion: Fix z and consider a smaller annulus $R_1' < |z - z_0| < R_2'$ that still contains z. Using Cauchy's formula, represent $f(z)$ as the sum of two integrals: one over circle $|z - z_0| = R_1'$, another over circle $|z - z_0| = R_2'$. Then use geometric series expansions.
To have a little shorter writing, you can take $z_0 = 0$.