HW3

1. Give an example of a real function \( y = f(x) \) (a real-valued function of a real variable) with these properties
   - the function is infinitely differentiable everywhere
   - its Taylor series at some point \( x_0 \) converges,
   - but it converges to a different function.

2. \{\*\} Find the radius of convergence of the Taylor series with the given center for the given function:
   - (a) \( f(z) = \frac{e^{z^2}}{(\sin z - 2)^2} \), \( z_0 = 1 \),
   - (b) \( f(z) = \frac{(z^2 - 1)^2}{(\cos z - 2)^3} \), \( z_0 = 1 \),
   - (c) \( f(x) = \frac{e^{-x^2}}{(x^2 + 1)^3} \) (\( x \) is a real variable), \( x_0 = 1 \),
   - (d) \( f(x) = e^{x^4} \) (\( x \) is a real variable), \( x_0 = 0 \),

(you do not need to find the series themselves).
3. \{\ast\}

(a) Prove the Liouville Theorem:
If \( f(z) \) is bounded and entire (i.e. analytic in the entire complex plane), then \( f(z) \) is constant.

\textit{Idea:} Consider Cauchy’s formula for \( f'(z) \), as integral over a “big” circle \(|\zeta - z| = R\). Then take the limit \( R \to \infty \) and show that \( f'(z) = 0 \).

(b) Prove the \textbf{Fundamental Theorem of Algebra}:
An \( n \)-th order polynomial \( P(z) \) has \( n \) roots in the complex plane.

\textit{Idea:} If \( P \) has no roots, then the function \( 1/P \) is bounded and entire.

(c) Prove: If \( f(z) \) is entire and grows at \( z \to \infty \) not faster than a linear function,
(i.e. there exist numbers \( A \) and \( B \), such that \( |f(z)| < A|z| + B \) for all complex \( z \)),
then \( f(z) \) is a linear function.

4. A real function \( y = f(x) \) is called \textit{real analytic} if it can be represented by a power series in the neighborhood of each point. Show that there is no analog of the Liouville Theorem for real analytic functions.
(a) Prove the **Morera Theorem:**
If \( f(z) \) is (1) continuous in a domain \( D \),
and (2) \( \oint_C f(z)dz = 0 \) for any closed curve \( C \) in \( D \),
then \( f(z) \) is analytic in \( D \).

* Idea: Fix a point \( z_0 \) and show that \( F(z) = \int_C f(z)dz \) (\( C \) is a curve from \( z_0 \) to \( z \)) is well defined and is an antiderivative of \( f(z) \).
There exists \( F''(z) \).

(b) Suppose, a simply-connected domain \( D \) consists of two disjoint domains \( D_1, D_2 \) and a curve \( \Gamma \), between them (\( D_1 \) and \( D_2 \) share a common boundary piece \( \Gamma \)); the function \( f(z) \) is analytic in \( D_1 \), analytic in \( D_2 \), and continuous in \( D \).
Show that \( f(z) \) is analytic in \( D \).

* Idea: To use the Morera Theorem, show that \( \oint_C f(z)dz = 0 \) for three kinds of closed curves \( C \):
(1) \( C \) is completely in \( D_1 \cup \Gamma \), (2) \( C \) is completely in \( D_2 \cup \Gamma \), and (3) \( C \) is partially in \( D_1 \) and partially in \( D_2 \).