HW3 **NAME: ....** **due 2/3/2016**

Only problerms marked by **{*}** are necessary to submit. You do not need to write the other problems if you know how to solve them. If in doubt (whether you can solve correctly or not), please write your solution, and I will check if it is correct.

1. Give an example of a real function $y = f(x)$ (a real-valued function of a real variable) with these properties
   - the function is infinitely differentiable everywhere
   - its Taylor series at some point x_0 converges,
   - but it converges to a different function.

2. **{*}** Find the radius of convergence of the Taylor series with the given center for the given function:

$$(a) \qquad f(z) = \frac{e^{z^2}}{(\sin z - 2)^2}, \qquad z_0 = 1,$$

$$(b) \qquad f(z) = \frac{(z^2-1)^2}{(\cos z - 2)^3}, \qquad z_0 = 1,$$

$$(c) \qquad f(x) = \frac{e^{-x^2}}{(x^2+1)^3} \quad (x \text{ is a real variable}), \qquad x_0 = 1,$$

$$(d) \qquad f(x) = e^{x^4} \quad (x \text{ is a real variable}), \qquad x_0 = 0,$$

   (you do not need to find the series themselves).

3. **{*}**

   (a) Prove the **Liouville Theorem:**
   If $f(z)$ is bounded and entire (i.e. analytic in the entire complex plane), then $f(z)$ is constant.

   Idea: Consider Cauchy's formula for $f'(z)$, as integral over a "big" circle $|\zeta - z| = R$. Then take the limit $R \to \infty$ and show that $f'(z) = 0$.

   (b) Prove the **Fundamental Theorem of Algebra:**
   An n-th order polynomial $P(z)$ has n roots in the complex plane.

   Idea: If P has no roots, then the function $1/P$ is bounded and entire.

   (c) Prove: If $f(z)$ is entire and grows at $z \to \infty$ not faster than a linear function,
   (i.e. there exist numbers A and B, such that $|f(z)| < A|z| + B$ for all complex z),
   than $f(z)$ *is* a linear function.

4. A real function $y = f(x)$ is called *real analytic* if it can be represented by a power series in the neighborhood of each point. Show that there is no analog of the Liouville Theorem for real analytic functions.

5. **{*}**

(a) Prove the **Morera Theorem:**
If $f(z)$ is (1) continuous in a domain D,
and (2) $\oint_C f(z)dz = 0$ for any closed curve C in D,
then $f(z)$ is analytic in D.

Idea: Fix a point z_0 and show that $F(z) = \int_C f(z)dz$ (C is a curve from z_0 to z) is well defined and is an antiderivative of $f(z)$. There exists $F''(z)$.

(b) Suppose, a simply-connected domain D consists of two disjoint domains $D_1,\ D_2$ and a curve Γ, between them (D_1 and D_2 share a common boundary piece Γ); the function $f(z)$ is analytic in D_1, analytic in D_2, and continuous in D.

Show that $f(z)$ is analytic in D.

Idea: To use the Morera Theorem, show that $\oint_C f dz = 0$ for three kinds of closed curves C:
(1) C is completely in $D_1 \cup \Gamma$, (2) C is completely in $D_2 \cup \Gamma$, and (3) C is partially in D_1 and partially in D_2.