HW2 **NAME: .....** **due 1/24/2018**

Homework contains "warm up" problems, which you do not need to write if you know how to solve them. Similar calculations will often come up as parts of other problems in the future. If in doubt (whether you can solve correctly or not), please write your solution, and I will check if it is correct. This assignment contains 5 problems (out of 7) that are necessary to submit; they are marked by **{*}**.

1. { "warm up" problem } We will see that if a complex function is differentiable (in complex sense) in a domain D, then it is *continuously* differentiable in D (and even infinitely differentiable, and even can be represented by its Taylor series).

   Give an example of real function $y = f(x)$ (x and y are real) which is differentiable everywhere, but not continuously differentiable.

2. **{*}** Give an example of real-valued function $u(x, y)$ of two real variables x and y that at some point (x_0, y_0) has the following properties:
   At this point, $u(x, y)$ has derivatives in all directions; all these directional derivatives are equal;
   but the function is not even continuous at (x_0, y_0).
   Is $u(x, y)$ differentiable at (x_0, y_0)?

3. **{*}** Find the values of four integrals

$$I = \int_C \bar{z} dz$$

along four curves C:

$$\begin{aligned}
&C_1 : \text{two segments from } z = 0 \text{ to } z = 1 \text{ to } z = 1 + i, \\
&C_2 : \text{one segment from } z = 0 \text{ to } z = 1 + i, \\
&C_3 : \text{arc from } z = 0 \text{ to } z = 1 + i \text{ of a circle with center at } z = 1, \\
&C_4 : \text{two segments from } z = 0 \text{ to } z = i \text{ to } z = 1 + i.
\end{aligned}$$

4. { "warm up" problem } Calculate

$$\oint_C z^n dz \qquad \text{for all integers } n = 0, \pm 1, \pm 2, \ldots,$$

where C is a circle $|z| = R$ (taken counterclockwise) with center at the origin and radius R.

5. **{*}** Evaluate the line integral

$$\int_C z^3 \, dz$$

over the path traversing from $z = 1 + i$ to $z = 1 - i$ along the following composite curve $C = C_1 + C_2 + C_3$

$$\begin{aligned} C_1 &: \quad \text{hyperbola } 2y^2 - x^2 = 1 \text{ from } z = 1 + i \text{ to } z = -2 + i\sqrt{5/2}, \\ C_2 &: \quad \text{segment } x = -2 \text{ from } z = -2 + i\sqrt{5/2} \text{ to } z = -2 - i\sqrt{5/2}, \\ C_3 &: \quad \text{hyperbola } 2y^2 - x^2 = 1 \text{ from } z = -2 - i\sqrt{5/2} \text{ to } z = 1 - i. \end{aligned}$$

6. **{*} Variations on the topic of Cauchy's Theorem.** Are the following statements true? No rigorous proof is required. Just tell what you think, and give a reason why you think so.

(a) If $f(z)$ is analytic in simply connected domain D and continuous in $\overline{D}$, then

$$\oint_{\partial D} f(z)dz = 0.$$

(b) Let $f(z)$ be analytic in a simply connected bounded domain D. Then for all curves C in D with common ends a and b, the integral

$$\int_C f(z)dz$$

has the same value (the integral depends only on the end points a, b and otherwise is independent of the integration path C).

(c) Let $f(z)$ be analytic in a simply connected domain D, and z_0 be an arbitrary point in D. Then the integral

$$F(z) = \int_{z_0}^{z} f(\zeta)d\zeta \qquad (z \in D)$$

(as a function of the upper integration limit) is also an analytic function in D, and

$$F'(z) = \frac{d}{dz}\int_{z_0}^{z} f(\zeta)dz = f(z).$$

(d) Let $f(z)$ be analytic in a domain D (D can be multiply connected, i.e. it can have holes). Then the value of the integral

$$I = \oint_C f(z)dz$$

remains unchanged if the closed curve C is continuously deformed, all the time being completely in D.

7. **{*}** Evaluate

$$\oint_C \frac{e^z}{z(z^2-16)}dz, \qquad \oint_C \frac{e^z}{z^2(z^2-16)}dz,$$

where C is circle $|z| = 3$.

I would like that you calculate these integrals without referring to the Residue Theorem, which we will consider later.
Suggestion: Deform C to a tiny circle $\Gamma : |z| = \epsilon$, and explain why

$$\oint_\Gamma \frac{f(z)}{z}dz = f(0)\oint_\Gamma \frac{1}{z}dz \qquad \text{if} \qquad f(z) = f(0) + O(z),\ z \to 0,$$

$$\oint_\Gamma \frac{f(z)}{z^2}dz = f'(0)\oint_\Gamma \frac{1}{z}dz \qquad \text{if} \qquad f(z) = f(0) + f'(0)z + O(z^2),\ z \to 0.$$