1. (a) For the function $u(x, y) = x + 4y$, find the function $v(x, y)$ such that $w = u + iv$ is an analytic function of the variable $z = x + iy$.

(b) For $v(x, y) = (1 + x)y$, find $u(x, y)$ such that $w = u + iv$ is an analytic function of the variable $z = x + iy$.

(c) Consider the function $u(x, y) = x^2$. Does there exist a function $v(x, y)$ such that $w = u + iv$ is an analytic function of the variable $z = x + iy$. 
2. We will see that if a complex function is differentiable (in complex sense) in a domain \( D \), then it is \textit{continuously} differentiable in \( D \) (and even infinitely differentiable, and even can be represented by its Taylor series).

Give an example of real function \( y = f(x) \) (\( x \) and \( y \) are real) which is differentiable everywhere, but not continuously differentiable.
3. Find the values of four integrals

\[ I = \int_C \bar{z} \, dz \]

along four curves \( C \):

\( C_1 \) : two segments from \( z = 0 \) to \( z = 1 \) to \( z = 1 + i \),
\( C_2 \) : one segment from \( z = 0 \) to \( z = 1 + i \),
\( C_3 \) : arc from \( z = 0 \) to \( z = 1 + i \) of a circle with center at \( z = 1 \),
\( C_4 \) : two segments from \( z = 0 \) to \( z = i \) to \( z = 1 + i \).
4. Calculate

\[ \oint_C z^n \, dz \quad \text{for all integers } n = 0, \pm 1, \pm 2, \ldots, \]

where \( C \) is a circle \( |z| = R \) (taken counterclockwise) with center at the origin and radius \( R \).
5. Evaluate the line integral
\[
\int_C z^3 \, dz
\]
over the path \( C \) traversing from \( z = 1 + i \) to \( z = 1 - i \) along the following composite curve (\( C = C_1 + C_2 + C_3 \))

\( C_1 \): hyperbola \( 2y^2 - x^2 = 1 \) from \( z = 1 + i \) to \( z = -2 + i\sqrt{\frac{5}{2}} \),

\( C_2 \): segment \( x = -2 \) from \( z = -2 + i\sqrt{\frac{5}{2}} \) to \( z = -2 - i\sqrt{\frac{5}{2}} \),

\( C_3 \): hyperbola \( 2y^2 - x^2 = 1 \) from \( z = -2 - i\sqrt{\frac{5}{2}} \) to \( z = 1 - i \).
6. **Variations on the topic of Cauchy’s Theorem.** Are the following statements true?

(a) If \( f(z) \) is analytic in simply connected domain \( D \) and continuous in \( \bar{D} \), then

\[
\oint_{\partial D} f(z)dz = 0
\]

*You can assume that \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) (the unit circle).

(b) Let \( f(z) \) be analytic in a simply connected domain \( D \). Then for all curves \( C \) in \( D \) with common ends \( a \) and \( b \), the integral

\[
\int_{C} f(z)dz
\]

has the same value (the integral depends only on the end points \( a, b \) and otherwise is independent of the integration path \( C \)).

(c) Let \( f(z) \) be analytic in a simply connected domain \( D \), and \( z_0 \) be an arbitrary point in \( D \). Then the integral

\[
F(z) = \int_{z_0}^{z} f(\zeta)d\zeta \quad (z \in D)
\]

(as a function of the upper integration limit) is also an analytic function in \( D \), and

\[
F'(z) = \frac{d}{dz} \int_{z_0}^{z} f(\zeta)d\zeta = f(z).
\]

(d) Let \( f(z) \) be analytic in a domain \( D \) (\( D \) can be multiply connected, i.e. it can have holes). Then the value of the integral

\[
I = \oint_{C} f(z)dz
\]

remains unchanged if the closed curve \( C \) is continuously deformed, all the time being completely in \( D \).
7. Evaluate

\[ \oint_C \frac{e^z}{z(z^2 - 16)} \, dz, \quad \oint_C \frac{e^z}{z^2(z^2 - 16)} \, dz, \]

where \( C \) is circle \( |z| = 3 \).

\textit{Suggestion:} Deform \( C \) to a tiny circle \( \Gamma : |z| = \epsilon \), and show that

\[
\oint_{\Gamma} \frac{f(z)}{z} \, dz = f(0) \oint_{\Gamma} \frac{1}{z} \, dz \quad \text{if} \quad f(z) = f(0) + O(z), \ z \to 0,
\]

\[
\oint_{\Gamma} \frac{f(z)}{z^2} \, dz = f'(0) \oint_{\Gamma} \frac{1}{z} \, dz \quad \text{if} \quad f(z) = f(0) + f'(0)z + O(z^2), \ z \to 0.
\]