1. Find the dispersion relation of Rossby waves, whose evolution is described by the equation

\[ \frac{\partial}{\partial t} (\psi - \Delta \psi) = \frac{\partial}{\partial x} \psi \]

[for the function \( \psi(x, y, t) \)] and show that the long time behavior leads to the group velocity.

(It is a vector, since there are two spatial coordinates \( x, y \) and, correspondingly, two wave numbers.)
2. Consider integral

\[ I(s) = \int_C \frac{e^{s(z^2-1)}}{z-1/2} \, dz \]

with large parameter \( s \to +\infty \); \( C \) is the vertical line \( \text{Im} \, z = 1 \), from \( z = 1 - i\infty \) to \( z = 1 + i\infty \). Keener, page 466.

(a) i. What is the saddle point \( z_0 \)?
   
   ii. What is the path \( C_0 \) of steepest descent from \( z_0 \)?
   
   iii. What is the path of steepest ascent from \( z_0 \)?

(b) Find the three-term asymptotic expansion of the integral.
3. Consider integral

\[ I(s) = \int_0^1 \ln(t) \, e^{ist} \, dt \quad (s \to +\infty). \]

(a) [As you can see, \( v(0) \neq v(1) \); so the deformation of the integration path to the steepest descent path is not straightforward.]

i. Is there a saddle point?
ii. Is there a point of stationary phase?
iii. What is the path \( C_1 \) of steepest descent from \( z = 0 \)?
iv. What is the path of steepest ascent from \( z = 0 \)?
v. What is the path \( C_2 \) of steepest descent from \( z = 1 \)?
vi. What is the path of steepest ascent from \( z = 1 \)?

(b) Find the three-term asymptotic expansion of the integral.
4. To find asymptotics of the integral
\[ I(s) = \int_C f(z) e^{s w(z)} \, dz, \quad s \to +\infty, \]
we deform path \( C \) to the steepest descent path \( v(z) \equiv \text{Im}[w(z)] = \text{const} \), that goes through the saddle point \( z_0 \), \( w'(z_0) = 0 \). Suppose \( w''(z_0) = 0 \) as well, but \( w'''(z_0) = A e^{i\alpha} \neq 0 \) (\( A, \alpha \) are real numbers, \( A > 0 \)).

(a) What are the directions of steepest descent from the point \( z_0 \)?
(b) What are the directions of steepest ascent from the point \( z_0 \)?

[In this case, the surface \( u(z) \equiv \text{Re}[w(z)] \) near \( z_0 \) is sometimes called “monkey saddle.”]