HW11 NAME: ................................................................. due 4/11/2018

1. Comment. The Laplace method is for finding asymptotic behavior of the integral

$$f(x) = \int_a^b g(t)e^{x\phi(t)}dt$$

with large parameter $x \to +\infty$ [g(t), \phi(t) are given real functions].

The main contribution to this integral comes from the neighborhood of point $t = c$ where $\phi(t)$ is maximal. Why?

To find asymptotic behavior of $f(x)$ beyond the leading term:

(a) make a change of variables to have a simple $\phi(t)$ (typically linear or quadratic in $t$);
(b) expand $g(t)$ near $t = c$;
(c) integrate term-by-term.

Problem. For instance,

$$f(x) = \int_0^\infty g(t)e^{-xt}dt$$

with large parameter $x \to +\infty$.

Prove Watson’s lemma: If the function $g$ is represented by its Taylor series

$$g(t) = \sum_{n=0}^{\infty} a_n t^n$$

(on some interval $0 \leq t \leq p$)

and $g(t)$ grows not faster than exponentially at $t \to \infty$ ($|g(t)| < Ke^{bt}$ for some constants $K$ and $b$; in particular, this condition ensures the convergence of the integral for large $x$), then

$$f(x) \sim \sum_{n=0}^{\infty} a_n \int_0^\infty t^n e^{-xt} dt \quad (x \to +\infty).$$

(The integrals here can be evaluated in terms of the gamma-function.)
2. Find a 3-term approximation to the integral

\[ I(x) = \int_0^1 e^{-xt} t^{-1/2} \, dt \quad (x \to +\infty). \]

*Suggestion:* \[ \int_0^1 = \int_0^\infty - \int_1^\infty. \] The approximation includes gauge functions \( \frac{1}{\sqrt{x}}, \frac{e^{-x}}{x}, \frac{e^{-x}}{x^2} \).
3. The Method of Stationary Phase. Find the leading behavior of the following integral as \( x \to +\infty \)

(a) \[ \int_{1/2}^{2} (1 + t)e^{ix\left(\frac{t}{x} - t\right)} \, dt \]

(b) \[ \int_{0}^{1} \tan t \, e^{i\pi t^4} \, dt \]  
   *Suggestion: Consider changing the integration variable.*