HW11 **NAME: .....** **due 4/11/2018**

1. *Comment.* **The Laplace method** is for finding asymptotic behavior of the integral

$$f(x) = \int_a^b g(t) e^{x\phi(t)} dt \quad \text{with large parameter} \quad x \to +\infty \qquad [g(t), \phi(t) \text{ are given real functions}].$$

The main contribution to this integral comes from the neighborhood of point $t = c$ where $\phi(t)$ is maximal. *Why?*

To find asymptotic behavior of $f(x)$ beyond the leading term:

(a) make a change of variables to have a simple $\phi(t)$ (typically linear or quadratic in t);

(b) expand $g(t)$ near $t = c$;

(c) integrate term-by-term.

Problem. For instance,

$$f(x) = \int_0^\infty g(t) e^{-xt} dt \quad \text{with large parameter} \quad x \to +\infty \ .$$

Prove **Watson's lemma:** If the function g is represented by its Taylor series

$$g(t) = \sum_{n=0}^{\infty} a_n t^n \quad (\text{on some interval } 0 \le t \le p)$$

and $g(t)$ grows not faster than exponentially at $t \to \infty$ ($|g(t)| < Ke^{bt}$ for some constants K and b; in particular, this condition ensures the convergence of the integral for large x), then

$$f(x) \sim \sum_{n=0}^{\infty} a_n \int_0^\infty t^n \, e^{-x\,t} \, dt \qquad (x \to +\infty).$$

(The integrals here can be evaluated in terms of the gamma-function.)

2. Find a 3-term approximation to the integral

$$I(x) = \int_0^1 e^{-x\,t}\, t^{-1/2}\, dt \quad (x \to +\infty).$$

Suggestion: $\int_0^1 = \int_0^\infty - \int_1^\infty$. The approximation includes gauge functions $\frac{1}{\sqrt{x}}$, $\frac{e^{-x}}{x}$, $\frac{e^{-x}}{x^2}$.

3. **The Method of Stationary Phase.** Find the leading behavior of the following integral as $x \to +\infty$

(a) $\int_{1/2}^{2} (1+t) e^{ix\left(\frac{t^3}{3} - t\right)}\, dt$

(b) $\int_0^1 \tan t \;\; e^{ixt^4}\, dt$ *Suggestion: Consider changing the integration variable.*