HW9 **NAME: ....** **due 3/30/2016**

1. **Discrete Fourier Transform DFT.** Consider the DFT of a vector $\mathbf{f} = \begin{pmatrix} f_0 \\ f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_{N-1} \end{pmatrix} \in \mathbb{C}^N$

$$F_k = \sum_{n=0}^{N-1} f_n \, e^{-i\,k\,\frac{2\pi}{N}\,n} \,.$$

(a) Show that the sequence $\{F_k\}$ is N-periodic: $F_k = F_{k+N}, \quad (-\infty < k < \infty)$.

[So, we can keep only N-component vector $\mathbf{F} = \begin{pmatrix} F_0 \\ F_1 \\ \cdot \\ \cdot \\ \cdot \\ F_{N-1} \end{pmatrix}$, and the DFT is a transformation in $\mathbb{C}^N$.]

(b) Show that the DFT is an expansion of $\mathbf{F} \in \mathbb{C}^N$ over vectors

$$\mathbf{v}_n = \begin{pmatrix} 1 \\ e^{-i\,\frac{2\pi}{N}\,n} \\ \cdot \\ \cdot \\ \cdot \\ e^{-i\,(N-1)\,\frac{2\pi}{N}\,n} \end{pmatrix} \qquad (n = 1, 2, \ldots, N-1).$$

Show that these vectors form an orthogonal basis in $\mathbb{C}^N$.

(c) Find the inverse transform, that restores f_n from the vector $\mathbf{F}$.

(d) Show that your formula for the inverse transform defines a periodic sequence $\{f_k\}$: $f_k = f_{k+N}$.

(e) Show that if $\mathbf{f}$ is real, then $F_{N-k} = \overline{F_k}$.

2. (a) Write out explicitely (w/o the exponent $e^{ik\frac{2\pi}{N}n}$) DFT for $N = 2$, and $N = 4$.

   (b) Convolution example. Let

$$\mathbf{f} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

   Find $\mathbf{h} = \mathbf{f} * \mathbf{g}$.

   (c) **Multiplication of polynomials as convolution**.
   Consider the multiplication of quadratic polynomials, giving a polynomial of degree 4:

$$(f_0 + f_1 x + f_2 x^2)(g_0 + g_1 x + g_2 x^2) = h_0 + h_1 x + h_2 x^2 + h_3 x^3 + h_4 x^4.$$

   Show that the cofficient vector $\mathbf{h}$ can be obtained as the convolution of the coefficient vectors $\mathbf{f}$ and $\mathbf{g}$ padded with zeros so that the length of all vectors is 5:

$$\mathbf{f} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{h} = \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix}.$$

   Namely, show that $\mathbf{h} = \mathbf{f} * \mathbf{g}$.

   (d) Show that **DFT turns convolution into multiplication**.
   Let $\mathbf{f}$, $\mathbf{g}$ $\in \mathbb{C}^N$ and $\mathbf{h} = \mathbf{f} * \mathbf{g}$. If $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$ have DFTs $\mathbf{F}, \mathbf{G}, \mathbf{H}$ $\in \mathbb{C}^N$ then $H_k = F_k G_k$.

   (e) **Parseval's identities**.

      i. Show that DFT preserves the inner product (up to normalization).

      ii. Show that DFT preserves the norm (up to normalization).

3. **Rouche's theorem**

(a) Prove: If the functions $f(z)$ and $\phi(z)$ are analytic inside and on simple closed curve C and if the strict inequality $|\phi(z)| < |f(z)|$ holds for all $z \in C$, then the function $f(z) + \phi(z)$ has exactly as many zeros (counting their multiplicities) inside C as the function $f(z)$.

Suggestion: First, show that the functions $f(z)$ and $f(z) + \phi(z)$ do not vanish on C.
Then show that $\Delta_C \arg\{f(z)\} = \Delta_C \arg\{f(z) + \phi(z)\}$

(b) Determine the number of roots (counting their multiplicities) of the equation $z^4 + 3z^3 + 6 = 0$ inside the circle $|z| = 2$.

(c) Determine the number of roots (counting their multiplicities) of the equation $2z^5 - 6z^2 + z + 1 = 0$ inside the annulus $1 \leq |z| \leq 2$.

4. (a) Let a continuously differentiable real function $y = f(x)$ take some interval (a, b) to an interval (α, β).
Suppose $f'(x) \neq 0$ for all $x \in (a, b)$.
Is it true that the map $y = f(x)$ is one-to-one?

(b) Let an analytic function $w = f(z)$ take some domain D to a domain Δ. Suppose, $f'(z) \neq 0$ for all $z \in D$.
Is it true that the map $w = f(z)$ is one-to-one?
What if D is simply connected?

(c) A continuously differentiable real function $y = f(x)$ takes one-to-one an interval (a, b) to an interval (α, β).
Does this imply $f'(x) \neq 0$ for all $x \in (a, b)$?

(d) An analytic function $w = f(z)$ takes a domain D to a domain Δ one-to-one.
Does this imply $f'(z) \neq 0$ for all $z \in D$?

5. (a) Show that all bilinear transformations form a *group*, i.e.
   i. the inverse of a bilinear map is also a bilinear map,
   ii. composition of two bilinear maps is bilinear.
   (b) Does the set of all conformal transformations form a group?

6. (a) Show that a bilinear map takes circles into circles.

Suggestion: Follow these steps:

i. Any bilinear map is a composition of linear maps and inversion.
ii. A linear map takes circles into circles.
iii. The inversion $w = 1/z$ takes circles into circles. [This is trivial for a circle centered at the origin. But it is true for any circle. To show this, write the equation of a circle in terms of z and $\bar{z}$.]

(b) Show that any circle in the z-plane can be mapped onto any circle in the w-plane by a suitable bilinear map.

Suggestion: Note that any circle is determined by three points (on its circumference) and consider the equation

$$\frac{(w_1 - w)(w_3 - w_2)}{(w_1 - w_2)(w_3 - w)} = \frac{(z_1 - z)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z)}.$$

See that it defines a bilinear transformation $w(z)$ which takes points z_1, z_2, z_3 to points w_1, w_2, w_3.