HW8 **NAME: ....** **due Wednesday, 3/23/2016**

1. Let $F(\mu)$ be the Fourier transform of $f(x)$; $-\infty < x < \infty,\ -\infty < \mu < \infty$.
   The smoother is $f(x)$, the faster $F(\mu)$ vanishes at ∞.
   The faster $f(x)$ vanishes at ∞, the smoother $F(\mu)$ is.

   (a) Prove: If $f(x)$ and its derivatives up to the order n are of the class $L^1(-\infty,\infty)$, then $F(\mu) = o(\mu^{-n}),\ \mu \to \infty$.

   (b) Prove: If the functions $f(x)$, $xf(x), \ldots, x^m f(x)$ are of the class $L^1(-\infty,\infty)$, then $F(\mu)$ has continuous derivatives up to the order m, and each of them vanishes at infinity.

   Suggestion: Differentiation in the x-space means multiplication in the Fourier space; and vice versa: multiplication of $f(x)$ by powers of x implies differentiation in the Fourier space. Use the Riemann-Lebesgue lemma.

   (c) Let
   $$f(x) = \begin{cases} 1, & |x| < \pi, \\ 0, & \text{otherwise.} \end{cases}$$
   How fast does $F(\mu)$ vanishes at ∞?

   (d) Let
   $$f(x) = \begin{cases} 1 + \cos x, & |x| < \pi, \\ 0, & \text{otherwise.} \end{cases}$$
   How fast does $F(\mu)$ vanishes at ∞?

   Suggestion: You can use MATHEMATICA or MAPLE to find the Fourier transforms.

2. **Computing Fourier transform.** Consider a signal containing a 50 Hz cosine of amplitude 0.7 and a 120 Hz sine of amplitude 1, corrupted with zero-mean white noise with a variance of 4

$$s(t) = 0.7\cos(2\pi\, 50\, t) + \sin(2\pi\, 120\, t) + r(t).$$

   (a) Chose time window (time duration) T and sampling rate F_s (time between samples is $1/F_s$). Take the Fourier transform of the signal and find the sine and cosine components hidden by the noise. Determine the frequency range and plot the Fourier transform.

   (b) What important frequencies and the corresponding Fourier amplitudes do you find? (The amplitudes are not exactly at $0.7/2$ and $1/2$, as expected. Why?) Change time window T to find the amplitudes with higher accuracy.

   Suggestion: A similar example is considered in MATLAB, "help" for command `fft` (fast Fourier transform). Unlike the MATLAB example, our signal is not even, and you might need the command `fftshift`.

3. Prove:

(a) If $f(z)$ has a zero at z_0 of order m, then

$$\operatorname{Res}\left[\frac{f'(z)}{f(z)}\,;\quad z=z_0\right] \;=\; m$$

(b) If $f(z)$ has a pole at z_0 of order n, then

$$\operatorname{Res}\left[\frac{f'(z)}{f(z)}\,;\quad z=z_0\right] \;=\; -n$$

4. (a) Suppose $f(z)$ is analytic inside and on a simple closed curve C, except for finite number of poles inside C, and moreover, $f(z) \neq 0$ on C.
Then

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz \quad = \quad N_0 - N_\infty \ ,$$

where N_0 and N_∞ are (respectively) the number of zeros and the number of poles of $f(z)$ inside C. Both zeros and poles are to be counted with their orders (multiplicities).

(b) **Argument Principle.**

$$N_0 - N_\infty \quad = \quad \frac{1}{2\pi} \Delta_C \arg\{f(z)\} \quad = \quad \left\{ \begin{array}{l} \text{the number of times} \\ \text{the point } w = f(z) \text{ surrounds the origin } w = 0 \\ \text{as } z \text{ describes } C \end{array} \right\}$$

(the conditions of the previous statement are to be satisfied;
$\Delta_C \arg\{f(z)\}$ denotes the total variation of $\arg\{f(z)\}$ as the point z describes the curve C).

5. (a) Using the Argument principle, determine the number of roots in the first quadrant for the equation

$$z^5 + 1 = 0.$$

(b) Prove the **fundamental theorem of algebra**, using the argument principle.