1. Calculate the DFT (with arbitrary $N$) of exponential sequence $f_n = e^{i\omega n}$; $n = 1, 2, \ldots, N-1$; $\omega$ is a real number. The continuous Fourier transform of this function is zero, besides one point. Is this true for the discrete Fourier transform? Distinguish two cases:

(a) The frequency $\omega$ is an integer multiple of $\frac{2\pi}{N}$ (in-bin sinusoid).

(b) The frequency $\omega$ is not an integer multiple of $\frac{2\pi}{N}$ (out-of-bin sinusoid).
2. Find a 3-term approximation to the integral

\[ I(x) = \int_0^1 e^{-xt} t^{-1/2} \, dt \quad (x \to +\infty). \]

*Suggestion:* \[ \int_0^1 = \int_0^\infty - \int_1^\infty. \] The approximation includes gauge functions \( \frac{1}{\sqrt{x}}, \frac{e^{-x}}{x}, \frac{e^{-x}}{x^2} \).
3. The Method of Stationary Phase.

(a) Find the leading behavior of the integral

\[ \int_{1/2}^{2} (1 + t)e^{ix\left(\frac{t^3}{3} - t\right)} \, dt \quad \text{as} \quad x \to +\infty. \]

(b) Find the leading behavior of the integral

\[ \int_{0}^{1} \tan t \, e^{ixt^4} \, dt \quad \text{as} \quad x \to +\infty. \]

(You might want to change the integration variable.)

(c) i. Find the dispersion relation of Rossby waves, whose evolution is described by the equation

\[ \frac{\partial}{\partial t}(\psi - \Delta \psi) = \frac{\partial}{\partial x} \psi \]

for the function \( \psi(x, y, t) \).

ii. Show that the long time behavior leads the **group velocity** and calculate it.

(It is a vector, since there are two spatial coordinates \( x, y \) and, correspondingly, two wave numbers.)
4. To find asymptotics of the integral

\[ I(s) = \int_C f(z) e^{sw(z)} \, dz, \quad s \to +\infty, \]

we deform path \( C \) to the steepest descent path, that goes through the saddle point \( z_0 \), \( w'(z_0) = 0 \). Suppose \( w''(z_0) = 0 \), but \( w'''(z_0) = Ae^{i\alpha} \neq 0 \) (\( A, \alpha \) are real numbers, \( A > 0 \)).

(a) What are the directions of steepest descent from the point \( z_0 \)?
(b) What are the directions of steepest ascent from the point \( z_0 \)?

Sometimes the surface \( u(z) = \text{Re}[w(z)] \) near \( z_0 \) is called the “monkey saddle”. Why?
5. Consider integral

\[ I(s) = \int_C \frac{e^{s(z^2-1)}}{z-1/2} \, dz \]

with large parameter \( s \to +\infty \); \( C \) is the vertical line \( \text{Im} \, z = 1 \), from \( z = 1 - i\infty \) to \( z = 1 + i\infty \). Keener, page 466.

(a) i. What is the saddle point \( z_0 \)?
   ii. What is the path \( C_0 \) of steepest descent from \( z_0 \)?
   iii. What is the path of steepest ascent from \( z_0 \)?

(b) Find the three-term asymptotic expansion of the integral.
6. Consider integral

\[ I(s) = \int_0^1 \ln(t) e^{ist} \, dt \quad (s \to +\infty). \]

(a) As you can see, \( v(0) \neq v(1) \); so the deformation of the integration path to the steepest descent path is not straightforward.

i. Is there a saddle point?

ii. Is there a point of stationary phase?

iii. What is the path \( C_1 \) of steepest descent from \( z = 0 \)?

iv. What is the path of steepest ascent from \( z = 0 \)?

v. What is the path \( C_2 \) of steepest descent from \( z = 1 \)?

vi. What is the path of steepest ascent from \( z = 1 \)?

(b) Find the three-term asymptotic expansion of the integral.