HW12 **NAME: .....** **due 4/20/2016**

1. Calculate the DFT (with arbitrary N) of exponential sequence $f_n = e^{i\omega n}$; $n = 1, 2, \ldots, N-1$; ω is a real number.
   The continuous Fourier transform of this function is zero, besides one point.
   Is this true for the discrete Fourier transform? Distinguish two cases:

   (a) The frequency ω is an integer multiple of $\frac{2\pi}{N}$ (in-bin sinusoid).

   (b) The frequency ω is *not* an integer multiple of $\frac{2\pi}{N}$ (out-of-bin sinusoid).

2. Find a 3-term approximation to the integral

$$I(x) = \int_0^1 e^{-xt}\, t^{-1/2}\, dt \qquad (x \to +\infty).$$

Suggestion: $\int_0^1 = \int_0^\infty - \int_1^\infty$. The approximation includes gauge functions $\frac{1}{\sqrt{x}}$, $\frac{e^{-x}}{x}$, $\frac{e^{-x}}{x^2}$.

3. **The Method of Stationary Phase.**

(a) Find the leading behavior of the integral

$$\int_{1/2}^{2} (1+t)e^{ix\left(\frac{t^3}{3}-t\right)}\, dt \qquad \text{as} \qquad x \to +\infty.$$

(b) Find the leading behavior of the integral

$$\int_{0}^{1} \tan t \;\; e^{ixt^4}\, dt \qquad \text{as} \qquad x \to +\infty.$$

(You might want to change the integration variable.)

(c) i. Find the dispersion relation of Rossby waves, whose evolution is described by the equation

$$\frac{\partial}{\partial t}(\psi - \Delta\psi) = \frac{\partial}{\partial x}\psi$$

for the function $\psi(x, y, t)$.

ii. Show that the long time behavior leads the **group velocity** and calculate it.
(It is a vector, since there are two spatial coordinates x, y and, correspondingly, two wave numbers.)

4. To find asymtotics of the integral

$$I(s) = \int_C f(z)\, e^{s\, w(z)}\, dz, \quad s \to +\infty,$$

we deform path C to the steepest descent path, that goes through the saddle point z_0, $w'(z_0) = 0$. Suppose $w''(z_0) = 0$, but $w'''(z_0) = Ae^{i\alpha} \neq 0$ (A, α are real numbers, $A > 0$).

(a) What are the directions of steepest descent from the point z_0 ?

(b) What are the directions of steepest ascent from the point z_0 ?

Sometimes the surface $u(z) = \mathrm{Re}[w(z)]$ near z_0 is called the "monkey saddle". Why?

5. Consider integral

$$I(s) = \int_C \frac{e^{s(z^2-1)}}{z-1/2}\, dz$$

with large parameter $s \to +\infty$; C is the vertical line $\operatorname{Im} z = 1$, from $z = 1 - i\infty$ to $z = 1 + i\infty$. Keener, page 466.

(a) i. What is the saddle point z_0?

ii. What is the path C_0 of steepest descent from z_0?

iii. What is the path of steepest ascent from z_0?

(b) Find the three-term asymptotic expansion of the integral.

6. Consider integral

$$I(s) = \int_0^1 \ln(t)\, e^{ist}\, dt \qquad (s \to +\infty).$$

(a) As you can see, $v(0) \neq v(1)$; so the deformation of the integration path to the steepest descent path is not straightforward.

i. Is there a saddle point?

ii. Is there a point of stationary phase?

iii. What is the path C_1 of steepest descent from $z = 0$?

iv. What is the path of steepest ascent from $z = 0$?

v. What is the path C_2 of steepest descent from $z = 1$?

vi. What is the path of steepest ascent from $z = 1$?

(b) Find the three-term asymptotic expansion of the integral.