HW10 **NAME: ....** **due Wednesday, 4/6/2016**

1. When applying a spectral method to a two-dimensional problem, we represent the unknown as a linear combination of basis functions, e.g.

$$f(x,y) = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F_{k,l}\, \phi_k(x)\, \psi_l(y).$$

which we consider on a two-dimensional grid $(x_m, y_n) \quad (m = 0, 1, \ldots, M-1;\ n = 0, 1, \ldots, N-1)$.
[Here we suppose that our domain is rectangle; $\phi(x)$ and $\psi(y)$ are basis functions in x and y directions respectively.]

   (a) How many operations (additions and multiplications) will the direct computation of the double sum

$$f(x_m, y_n) = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F_{k,l}\, \phi_k(x_m)\, \psi_l(y_n) \qquad [m = 0, 1, \ldots, M-1;\ n = 0, 1, \ldots, N-1]$$

   require?

   (b) One can significantly reduce the computational costs by rearranging the double sum into two consecutive sums

$$f(x_m, y_n) = \sum_{k=0}^{M-1} \phi_k(x_m) \left\{ \sum_{l=0}^{N-1} F_{k,l}\, \psi_l(y_n) \right\} \qquad [m = 0, 1, \ldots, M-1;\ n = 0, 1, \ldots, N-1].$$

   How many operations will this computation require?

   Note: The values of the basis functions $\phi_k(x)$, $\psi_l(y)$ at the grid points x_m, y_n can be pre-computed, while the sums are computed many times (at each time-step); so, the computation of $\phi_k(x_m)$, $\psi_l(y_n)$ should be excluded from the computational cost in (a) and (b).

   (c) Consider the two dimensional FFT

$$F_{k,l} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_{m,n}\, e^{i(k\frac{2\pi}{M}m + l\frac{2\pi}{N}n)}$$

   acting on the $N \times M$ matrix $[f_{n,m}]$ of samples $f_{m,n} = f(x_m, y_n)$. It is usually realized by computing the one-dimensional FFTs of each column, then of each row of the result.
   How many operations will this computation require?

2. **Computer simulation of non-linear PDEs.**

As an example, consider PDE

$$\frac{\partial}{\partial t}(\Delta\psi - \psi) + \psi_x + \psi_x \Delta\psi_y - \psi_y \Delta\psi_x = 0.$$

Several essentially different physical problems — the dynamics of atmosphere and ocean, nuclear fusion with magnetic confinement, magnetohydrodynamics of Earth's liquid iron core — lead to this equation.

To simulate this equation numerically, we consider it in a box ($0 \le x \le L_x$, $0 \le y \le L_y$ with periodic boundary conditions), represent the unknown function as superposition of Fourier harmonics

$$\psi(x, y, t) = \sum_{k=-M_x}^{M_x} \sum_{l=-M_y}^{M_y} \psi_{k,l}(t)\, e^{i(k\frac{2\pi}{L_x}x + l\frac{2\pi}{L_y}y)},$$

and reduce our PDE to a system of ODEs for the Fourier coefficients $\psi_{k,l}(t)$. We then integrate this system in time t using the standard ODE solver, e.g. Runge-Kutta method.

(a) Write out the ODE system.

(b) How would you compute the nonlinear terms? The goal is speed, avoiding "aliasing".

3. Prove **the Schwartz Lemma:** Let $f(z)$ be

- analytic in $U = \{z : |z| < 1\}$,
- $|f(z)| \leq 1$ in U,
- $f(0) = 0$.

Then $|f(z)| \leq |z|$ in U.

Suggestion: Consider function $h(z) = \frac{f(z)}{z}$. Show that it is analytic in U (the singularity at $z = 0$ can be removed). Take an arbitrary $\rho \in (0, 1)$. Apply the maximum principle to $h(z)$ to show that $|h(z)| \leq 1/\rho$ when z is in the disc $U_\rho = \{z : |z| \leq \rho\}$. Finally, take the limit as $\rho \to 1$. Why do you need to consider a smaller disc $U_\rho \subset U$?

4. Show that **a 1-to-1 conformal map of a disc onto a disc is necessarily bilinear**.

Suggestion: Consider an arbitrary disc in the ζ-plane and another arbitrary disc in the ω-plane. Riemann says that there is a 1-to-1 conformal transformation of the first disc onto the second one; you need to show that this conformal map is bilinear. Follow these steps:

(a) Some point ζ_0 of the first disc is taken to some ω_0 of the second disc.
There is a bilinear transformation $\zeta \to \tilde{\zeta}$ of the ζ-plane disc onto the unit disc $\{|\tilde{\zeta}| < 1\}$ (HW9). Herewith ζ_0 is taken to some $\tilde{\zeta}_0$.
Similar, there is a bilinear transformation $\omega \to \tilde{\omega}$ of the ω-plane disc onto the unit disc $\{|\tilde{\omega}| < 1\}$. Herewith ω_0 is taken to some $\tilde{\omega}_0$.

(b) Find the bilinear transformation that maps $\{|\tilde{\zeta}| < 1\}$ onto the unit disc $U^z = \{|z| < 1\}$ and takes $\tilde{\zeta}_0$ to the origin.
Similar, find the bilinear transformation that maps $\{|\tilde{\omega}| < 1\}$ onto $U^w = \{|w| < 1\}$ and takes $\tilde{\omega}_0$ to the origin.

(c) Use the Schwartz lemma to show that if a conformal transformation $w = f(z)$ maps U^z onto U^w so that the origin of z-plane is taken to the origin of the w-plane, then $f(z) = e^{i\beta}z$ (β is a real number).
Indeed, according to the Schwartz lemma, $|w| \le |z|$. The Schwartz lemma for the inverse transformation $z = g(w)$ says $|z| \le |w|$. So, $\left|\frac{f(z)}{z}\right| = 1$ for all $z \in U^z$. If the absolute value of an analytic function is constant, then the function is a constant (HW4).

(d) Use the group property (HW9) to show that the original transformation of the ζ-plane disc onto the ω-plane disc is *bilinear*.

5. Solve Laplace's equation

$$\phi_{xx} + \phi_{yy} = 0 \quad \text{in the domain between two circles} \quad x^2 + y^2 = 1 \quad \& \quad (x-1)^2 + y^2 = \left(\frac{5}{2}\right)^2$$

subject to the boundary condition

$$\phi(x,y) = a \text{ when } x^2 + y^2 = 1 \quad \text{and} \quad \phi(x,y) = b \text{ when } (x-1)^2 + y^2 = \left(\frac{5}{2}\right)^2$$

(a and b are real parameters).

Suggestion: Conformally transform the domain between the two non-concentric circles onto a domain between concentric circles and use new symmetry to reduce the PDE to an ODE.

Such transformation is necessarily bilinear; the map

$$\zeta = e^{i\beta}\frac{z-m}{1-\bar{m}z}$$

maps the unit disc to itself. We can use the three parameters m and β to map the external circle into some circle with its center at the origin and some radius R (undetermined constant). We can take $\beta = 0$ and real m, so that we have a single real parameter; we find m solving a certain quadratic equation.