ABSTRACT. We consider nonlinear integral equations of Fredholm and Volterra type with respect to functions having values in $L$-spaces. Such class of equations includes set-valued integral equations, fuzzy integral equations and many others. We prove theorems of existence and uniqueness of the solutions for such equations and investigate data dependence of their solutions.

1. Introduction. A wide variety of questions lead to Fredholm and Volterra integral equations. They have many important applications in biology, physics, and engineering (see, for example, [5], [14], [10], [9], and the references therein). Nowadays scientists are more and more interested in integral equations for functions with values that are compact and convex sets in finite or infinite dimensional spaces, or that are fuzzy sets, see [7], [4], [17], [13], [18], [15], [16], [3], [20]. In this paper we consider a generalized concept, that of an $L$-space, that encompasses all of these as special cases. In particular, we investigate the existence and uniqueness of solutions of nonlinear Fredholm integral equations and nonlinear Volterra integral equations of the second kind, for functions with values in $L$-spaces. Theorems of existence and uniqueness for linear Fredholm and Volterra integral equations for functions with values in $L$-spaces are obtained in [2]. We also investigate the dependence of solutions of such equations on variations of the data.

The paper is organized as follows. In Section 2 we list some preliminary results that will be used in the remainder of the paper.
In Section 3 we show existence and uniqueness of the solutions of these integral equations. Section 4 is devoted to questions of data dependence. We end the paper with a discussion in Section 5.

2. Preliminary Results.

2.1. L-spaces. The following definition was introduced in [19]:

Definition 2.1. A complete separable metric space $X$ with metric $\delta$ is said to be an $L-$ space if in $X$ operations of addition of elements and their multiplication with real numbers are defined, and the following axioms are satisfied:

$A1. \forall x, y \in X \quad x + y = y + x;$

$A2. \forall x, y, z \in X \quad x + (y + z) = (x + y) + z;$

$A3. \exists \theta \in X \forall x \in X \quad x + \theta = x \ (where \ \theta \ \text{is called a zero in} \ X);$

$A4. \forall x, y \in X \quad \lambda \in \mathbb{R} \quad \lambda(x + y) = \lambda x + \lambda y;$

$A5. \forall x \in X \quad \lambda, \mu \in \mathbb{R} \quad \lambda(\mu x) = (\lambda \mu)x;$

$A6. \forall x \in X \quad 1 \cdot x = x, \ 0 \cdot x = \theta;$

$A7. \forall x, y \in X \quad \lambda \in \mathbb{R} \quad \delta(\lambda x, \lambda y) = |\lambda|\delta(x, y);$}

$A8. \forall x, y, u, v \in X \quad \delta(x + y, u + v) \leq \delta(x, u) + \delta(y, v).$

Let us list some examples of $L$-spaces:

(1) Any Banach space $(Y, \| \cdot \|_Y)$ over the field of real numbers endowed with the metric $\delta(x, y) = \|x - y\|_Y$ is an $L$-space.

(2) Let $K(\mathbb{R}^n)$ be the set of all nonempty and compact subsets of $\mathbb{R}^n$ and let $C(\mathbb{R}^n) \subset K(\mathbb{R}^n)$ be the subset of convex sets. We define the required operations and the Hausdorff metric on $K(\mathbb{R}^n)$ as follows: For $A, B \in K(\mathbb{R}^n)$, and $\alpha \in \mathbb{R}$:

$A + B := \{x + y : x \in A, y \in B\} \ \ \ \ \ \ \ \ alpha A := \{\alpha x : x \in A\}.$
\[ \delta^h(A, B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} |x - y|, \sup_{x \in B} \inf_{y \in A} |x - y| \right\}, \]

where \(| \cdot |\) is the Euclidean norm in \(\mathbb{R}^n\). With these operations and metric, \(K(\mathbb{R}^n)\) and its subspace \(K^c(\mathbb{R}^n)\) are complete, separable metric spaces (see for example [8]) and since the axioms A1-A8 hold, these spaces are \(L\)-spaces.

(3) The set of all closed bounded subsets of a given Banach space, endowed with the Hausdorff metric, is an \(L\)-space.

(4) Any quasilinear normed space \(Y\) (definition see in [1]) is an \(L\)-space.

(5) Consider (see e.g., [6]) the class of fuzzy sets \(E^n\) consisting of functions \(u : \mathbb{R}^n \to [0, 1]\) such that

(a) \(u\) is normal, i.e. there exists an \(x_0 \in \mathbb{R}^n\) such that \(u(x_0) = 1\);

(b) \(u\) is fuzzy convex, i.e. for any \(x, y \in \mathbb{R}^n\) and \(0 \leq \lambda \leq 1\),

\[ u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}; \]

(c) \(u\) is upper semicontinuous;

(d) the closure of \(\{x \in \mathbb{R}^n : u(x) > 0\}\), denoted by \([u]^0\), is compact.

For each \(0 < \alpha \leq 1\), the \(\alpha\)-level set \([u]^\alpha\) of a fuzzy set \(u\) is defined as

\[ [u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\}. \]

The addition \(u + v\) and scalar multiplication \(cu, c \in \mathbb{R} \setminus \{0\}\), on \(E^n\) are defined, in terms of \(\alpha\)-level sets, by

\[ [u + v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [cu]^\alpha = c[u]^\alpha \text{ for each } 0 < \alpha \leq 1. \]

Define also \(0 \cdot u\) by the equality \([0 \cdot u]^\alpha = \{\theta\}\) (here \(\theta = (0, \ldots, 0) \in \mathbb{R}^n\)).

One of the possible metrics in \(E^n\) is defined in the following way. For a given \(1 \leq p < \infty\)

\[ d_p(u, v) = \left( \int_0^1 \delta([u]^\alpha, [v]^\alpha)^p d\alpha \right)^{1/p}. \]

Then the space \((E^n, d_p)\) is (see [6, Theorem 3]) a complete separable metric space and therefore an \(L\)-space.
2.2. Integrals of functions with values in $L$-spaces. We need the following notion of convex elements of $L$-space:

**Definition 2.2.** An element $x \in X$ is convex if
$$\forall \lambda, \mu \geq 0 \quad \lambda x + \mu x = (\lambda + \mu)x.$$  

Let $X^c$ be a set of all convex elements of a given $L$-space $X$.

**Remark 2.3.** $X^c$ is a closed subset of $X$.

We also need the definition of a convexifying operator (see [19]) which we give in a somewhat modified form.

**Definition 2.4.** Let $X$ be an $L$-space. The operator $P : X \to X^c$ is called a convexifying operator if

1. $\forall x, y \in X \quad \delta(P(x), P(y)) \leq \delta(x, y)$;
2. $P \circ P = P$;
3. $P(\alpha x + \beta y) = \alpha P(x) + \beta P(y), \forall x, y \in X, \alpha, \beta \in \mathbb{R}$.

Examples of convexifying operators include

1. The identity operator in the space $K^c(\mathbb{R}^n)$ is a convexifying operator.
2. The operator, that on the space $K(\mathbb{R}^n)$ is defined by the formula $P(A) = \text{co}(A)$, is a convexifying operator. Here by the $\text{co}(A)$ we denote the convex hull of a set $A$.
3. The identity operator in the space $(E^n, d_p)$ is a convexifying operator.

Below we discuss $L$-spaces $X$ with some fixed convexifying operator $P$. Denote the image of the element $x \in X$ for the mapping $P : X \to X$ as $\tilde{x}$ i.e. $Px = \tilde{x}, \forall x \in X$. If all elements of an $L$-space $X$ are convex in the sense of Definition 2.2, then we choose the identity operator as the convexifying operator.

Next we define the Riemannian integral for a function $f : [a, b] \to X$, where $X$ is an $L$-space. We again follow Vahrameev [19] for this purpose. Let $\tilde{f}(t) := f(t)$.
**Definition 2.5.** Let $I$ be an interval $[a, b]$. The mapping $f : I \to X$ is called weakly bounded, if $\delta(\theta, \tilde{f}(t)) \leq \text{const}$ and weakly continuous, if $\tilde{f} : I \to X$ is continuous.

**Remark 2.6.** Note that if a function $f : I \to X$ is continuous then $f$ is weakly continuous. If all elements $x \in X$ are convex (i.e. $P = \text{Id}$), then the concepts of continuity and weak continuity coincide.

We need the notion of a stepwise mapping from $I$ to an $L$-space $X$.

**Definition 2.7.** The mapping $f : I \to X$ is called stepwise, if there exist a set $\{x_k\}_{k=0}^n \subset X$ and a partition $a = t_0 < t_1 < ... < t_n = b$ of the interval $I$, such that $\tilde{f}(t) = \tilde{x}_k$ for $t_{k-1} < t < t_k$.

**Definition 2.8.** The Riemannian integral of a stepwise mapping $f : I \to X$ is an element of the space $X$ that is defined by the following equality:

$$
\int_a^b f(t) dt = \sum_{k=0}^{n-1} (t_{k+1} - t_k) \tilde{x}_k.
$$

**Definition 2.9.** We say that a weakly bounded mapping $f : I \to X$ is integrable in the Riemannian sense if there exist a sequence $\{f_k\}$ of stepwise mappings from $I$ to $X$, such that

$$
\int^* \delta(\tilde{f}(t), \tilde{f}_k(t)) dt \to 0, \quad \text{as } k \to \infty,
$$

where $f^*$ is a regular Riemannian integral for real-valued functions.

It follows from (2.1) that the sequence $\left\{ \int_a^b f_k(t) dt \right\}$ is a Cauchy sequence and thus we can use the following definition.

**Definition 2.10.** Let $f : I \to X$ be integrable in the Riemannian sense and let $\{f_k\}$ be a sequence of stepwise mappings such that (2.1) holds. Then the Riemannian integral of $f$ is the limit

$$
\int_a^b f(t) dt = \lim_{k \to \infty} \int_a^b f_k(t) dt.
$$
As described in [19] the Riemannian integral for a function \( f : I \rightarrow X \) has the following properties:

1. If \( f \) and \( g \) are integrable, then \( \forall \alpha, \beta \in \mathbb{R}, \ \alpha f + \beta g \) is integrable, and moreover
   \[
   \int_{a}^{b} (\alpha f(t) + \beta g(t)) \, dt = \alpha \int_{a}^{b} f(t) \, dt + \beta \int_{a}^{b} g(t) \, dt.
   \]

2. If \( f \) is integrable in the Riemannian sense, then \( \tilde{f} \) is also integrable and
   \[
   \int_{a}^{b} f(t) \, dt = \int_{a}^{b} \tilde{f}(t) \, dt.
   \]

3. If \( f \) and \( g \) are integrable, then the function \( t \rightarrow \delta(\tilde{f}(t), \tilde{g}(t)) \) is integrable in the Riemannian sense and
   \[
   \delta \left( \int_{a}^{b} f(t) \, dt, \int_{a}^{b} g(t) \, dt \right) \leq \int_{a}^{b} \delta \left( \tilde{f}(t), \tilde{g}(t) \right) \, dt.
   \]

The following theorem (see [19], [1]) guarantees that we can consider the integrals which arise below as Riemannian integrals.

**Theorem 2.11.** A weakly bounded mapping \( f : I \rightarrow X \) is integrable in the Riemannian sense if and only if it is weakly continuous almost everywhere on \( I \).

3. **Theorems of existence and uniqueness.** For functions with values in an \( L \)-space we consider integral equations

   \[ x(t) = f(t) + \lambda \int_{a}^{b} g(t, s, x(s)) \, ds \quad \text{Fredholm Equation} \]

   and

   \[ x(t) = f(t) + \int_{a}^{t} g(t, s, x(s)) \, ds \quad \text{Volterra Equation} \]

   and prove for these equations theorems of existence and uniqueness of their solutions. Our results generalize the results of I. Tişe [18] for the case of \( X = K^{c}(\mathbb{R}^{n}) \) and \( f(t) = A, \ A \in K^{c}(\mathbb{R}^{n}) \).
3.1. Fredholm equation. Consider the set \( Y = [a, b] \times [a, b] \times X \). In the space \( Y \) introduce a metric assuming that for points \( y = (t, s, x) \) and \( y' = (t', s', x') \) from \( Y \)

\[
d(x, y) = |t - t'| + |s - s'| + \delta(x, x').
\]

Consider the Fredholm integral equation

\[
(3.1) \quad x(t) = f(t) + \lambda \int_a^b g(t, s, x(s)) ds
\]

where \( f : [a, b] \to X \) and \( g : Y \to X \) are known functions, \( \lambda \) is fixed
real parameter and \( x : [a, b] \to X \) is an unknown function.

Theorem 3.1. Suppose the function \( f \) is continuous on \([a, b]\) and the function \( g(t, s, x) \) satisfies the following conditions

1. \( g \) is weakly continuous on \( Y \), so the function \( \tilde{g} : Y \to X^c \) is continuous;
2. there exists a constant \( K \) such that for \( \forall (t, s) \in [a, b] \times [a, b] \) the function \( \tilde{g} \) satisfies the Lipschitz condition with constant \( K > 0 \) on the variable \( x \), so \( \forall x', x'' \in X \)

\[
(3.2) \quad \delta(\tilde{g}(t, s, x'), \tilde{g}(t, s, x'')) \leq K \delta(x', x'').
\]

Then if \( |\lambda| < \frac{1}{K(b-a)} \) the equation (3.1) has a unique solution \( x \in C([a, b], X) \).

Proof. Denote by \( C([a, b], X) \) the space of continuous functions \( x : [a, b] \to X \). Introduce in this space a metric

\[
\rho(x, y) = \max_{t \in [a,b]} \delta(x(t), y(t)).
\]

It is known (see for example [1]) that the obtained space is complete and separable.

We consider an operator \( A \) on the space \( C([a, b], X) \) defined by

\[
(3.3) \quad Ax(t) := f(t) + \lambda \int_a^b g(t, s, x(s)) ds.
\]

Next we show that \( \forall x \in C([a, b], X) \) \( Ax \in C([a, b], X) \). For this it is enough to prove continuity of the operator \( Bx(t) = \int_a^b g(t, s, x(s)) ds \).
Let the function \( x \) be given. Consider a set 
\[ M = \{(t, s, x(s)) : t, s \in [a, b]\} \subset Y. \]

Due to the continuity of the function \( x \) this set is a compact subset of the space \( Y \). Constriction of the function \( \tilde{g} \) on \( M \) is a continuous function on \( M \) and thus is uniformly continuous.

Take an arbitrary \( \varepsilon > 0 \) and choose \( \eta > 0 \) such that
\[ \delta(\tilde{g}(t, s, x(s)), \tilde{g}(t', s', x(s'))) < \eta \]
\[ \Rightarrow \quad \frac{\varepsilon}{b - a}. \]

Estimate
\[ \delta(Bx(t'), Bx(t'')) = \delta \left( \int_a^b \tilde{g}(t', s, x(s))ds, \int_a^b \tilde{g}(t'', s, x(s))ds \right) \]
\[ \leq \int_a^b \delta(\tilde{g}(t', s, x(s)), \tilde{g}(t'', s, x(s)))ds. \]

Taking into account (3.4) we have that if \( |t' - t''| < \eta \), then
\[ \delta(Bx(t'), Bx(t'')) < \varepsilon. \]

Therefore the function \( Bx(t) \) is continuous.

Next we show that with \( |\lambda| < \frac{1}{K(b - a)} \) the operator \( A \) is contractive. We have
\[ \rho(Ax, Ay) = \max_{t \in [a, b]} \delta(Ax(t), Ay(t)) \]
\[ = \max_{t \in [a, b]} \delta \left( f(t) + \lambda \int_a^b g(t, s, x(s))ds, \right) \]
\[ f(t) + \lambda \int_a^b g(t, s, y(s))ds \]
\[ \leq \max_{t \in [a, b]} \delta \left( \lambda \int_a^b \tilde{g}(t, s, x(s))ds, \lambda \int_a^b \tilde{g}(t, s, y(s))ds \right) \]
\[ \leq |\lambda| \max_{t \in [a, b]} \int_a^b \delta(\tilde{g}(t, s, x(s)), \tilde{g}(t, s, y(s)))ds. \]

Due to the third condition of the Theorem 3.1 \( \forall t, s: \)
\[ \delta(\tilde{g}(t, s, x(s)), \tilde{g}(t, s, y(s))) \leq K \delta(x(s), y(s)). \]

Therefore,
\[ \rho(Ax, Ay) \leq |\lambda| \int_a^b K \delta(x(s), y(s))ds \]
\[ \leq |\lambda| K \max_{s \in [a, b]} \delta(x(s), y(s)) \int_a^b ds = |\lambda| K(b - a) \rho(x, y). \]
If $|\lambda| < \frac{1}{K(b-a)}$ then this operator is contractive and thus the equation (3.1) has a unique solution.

Remark 3.2. Note that if the known function $f$ in (3.1) is convex-valued ($f : [a, b] \to X^c$), then the solution of equation (3.1) also is convex-valued.

3.2. Volterra Equation. Consider the set $Y = [a, b] \times [a, b] \times X$. The Volterra integral equation has the following form

\[
(3.5) \quad x(t) = f(t) + \int_a^t g(t, s, x(s))ds
\]

where $x : [a, b] \to X$ is an unknown function, $g : Y \to X$ is a known function, and $f : [a, b] \to X$ is a known function.

Theorem 3.3. Suppose the function $f$ is continuous on $[a, b]$ and the function $g(t, s, x)$ satisfies the following conditions

1. $g$ is weakly continuous on $Y$, so the function $\tilde{g} : Y \to X^c$ is continuous.
2. There exist a constant $K$ such that for $\forall (t, s) \in [a, b] \times [a, b]$ the function $\tilde{g}$ satisfies the Lipschitz condition (3.2) with a constant $K$ on variable $x$.

Then the equation (3.5) has a unique solution $x \in C([a, b], X)$.

Proof. Consider an operator $A : C([a, b], X) \to C([a, b], X)$:

\[
Ax(t) := f(t) + \int_a^t g(t, s, x(s))ds
\]

(That fact that $Ax \in C([a, b], X)$ if $x \in C([a, b], X)$ can be derived analogously to how it was done in the previous section). We have

\[
\delta(Ax(t), Ay(t)) = \delta \left( f(t) + \int_a^t g(t, s, x(s))ds, f(t) + \int_a^t g(t, s, y(s))ds \right)
\]

\[
\leq \delta \left( \int_a^t g(t, s, x(s))ds, \int_a^t g(t, s, y(s))ds \right)
\]

\[
\leq \int_a^t \delta(\tilde{g}(t, s, x(s)), \tilde{g}(t, s, y(s)))ds.
\]
Using the second condition of the Theorem 3.3 we obtain
\begin{equation}
\delta(Ax(t), Ay(t)) \leq \int_a^t K\delta(x(s), y(s))ds \leq K \max_{s \in [a,b]} \delta(x(s), y(s)) \int_a^t ds
= K(t - a)\rho(x, y).
\end{equation}
Therefore,
\[ \rho(Ax, Ay) \leq K(b - a)\rho(x, y), \]
and thus if \( b - a < \frac{1}{K} \) then this operator is contractive and on any interval \([a, b]\), where \( 0 < b - a < \frac{1}{K} \) the equation (3.5) has a unique solution.

We now prove by induction, that \( \forall n \geq 1 \)
\begin{equation}
\delta(A^n x(t), A^n y(t)) \leq \frac{(t - a)^n}{n!} K^n \rho(x, y).
\end{equation}
Inequality (3.6) is the induction base case. Assume that
\[ \delta(A^{n-1} x(t), A^{n-1} y(t)) \leq \frac{(t - a)^{n-1}}{(n-1)!} K^{n-1} \rho(x, y). \]
Then
\begin{align*}
\delta(A^n x(t), A^n y(t)) &= \delta \left( f(t) + \int_a^t g(t, s, A^{a-1} x(s))ds, \\
&\quad f(t) + \int_a^t g(t, s, A^{a-1} y(s))ds \right) \\
&\leq \delta \left( \int_a^t \bar{g}(t, s, A^{a-1} x(s))ds, \\
&\quad \int_a^t \bar{g}(t, s, A^{a-1} y(s))ds \right) \\
&\leq \int_a^t \delta \left( \bar{g}(t, s, A^{a-1} x(s)), \bar{g}(t, s, A^{a-1} y(s)) \right) ds \\
&\leq \int_a^t K \delta \left( A^{a-1} x(s), A^{a-1} y(s) \right) ds \\
&\leq \int_a^t K (s-a)^{n-1} K^{n-1} \rho(x, y)ds \\
&= \frac{K^n}{(n-1)!} \rho(x, y) \int_a^t (s-a)^{n-1} ds \\
&= \frac{(t-a)^n}{n!} K^n \rho(x, y).
\end{align*}
Therefore inequality (3.7) is proved. It implies that for any \( a < b \) and any \( n \)
\[ \rho(A^n x, A^n y) = \max_{a \leq t \leq b} \delta(A^n x(t), A^n y(t)) \leq \frac{(b-a)^n}{n!} K^n \rho(x, y). \]
If \( n \) is sufficiently large, then \( \frac{(b-a)^n}{n!} K^n < 1 \), and therefore \( A^n \) is contractive operator. Using the generalized contractive mapping principle (see [12, ch.2 §14]) we see that the operator \( A \) has a unique fixed point. Thus the equation (3.5) has a unique solution in the space \( C([a, b], X) \).

\[ \square \]

4. Data Dependence. In this section we consider questions about the dependence of solutions of equations (3.1) and (3.5) on perturbations of the given functions \( g(t, s, x) \) and \( f(t) \). These questions were considered in [18] for set-valued functions. We show that the schemes of the proof of Theorems 3.2 and 3.4 from [18] work in our more general setting.

**Theorem 4.1.** Consider the set \( Y = [a, b] \times [a, b] \times X \) and let \( g_1, g_2 : Y \to X \) be weakly continuous. Consider also the following equations:

\[(4.1) \quad x(t) = f_1(t) + \int_a^b g_1(t, s, x(s))ds,\]

\[(4.2) \quad y(t) = f_2(t) + \int_a^b g_2(t, s, y(s))ds.\]

Suppose:

1. For any \((t, s) \in [a, b] \times [a, b]\) the function \( \tilde{g}(t, s, x) \) satisfies Lipschitz condition (3.2) on variable \( x \) and \( K(b - a) < 1 \). Denote by \( x^*(t) \) the unique solution of the equation (4.1).
2. There exist \( \eta_1, \eta_2 > 0 \) such that \( \delta(\tilde{g}_1(t, s, x), \tilde{g}_2(t, s, x)) \leq \eta_1 \) for all \((t, s, x) \in [a, b] \times [a, b] \times X \) and \( \rho(f_1(t), f_2(t)) \leq \eta_2 \).
3. There exists \( y^*(t) \) a solution of the equation (4.2).

Then

\[ \rho(x^*, y^*) \leq \frac{\eta_2 + \eta_1(b - a)}{1 - K(b - a)}. \]
**Proof.** We have
\[
\delta(x^*(t), y^*(t)) = \delta \left( f_1(t) + \int_a^b \tilde{g}_1(t, s, x^*(s))ds, \
\quad f_2(t) + \int_a^b \tilde{g}_2(t, s, y^*(s))ds \right) \\
\leq \delta \left( f_1(t), f_2(t) \right) \\
+ \delta \left( \int_a^b \tilde{g}_1(t, s, x^*(s))ds, \int_a^b \tilde{g}_1(t, s, y^*(s))ds \right) + \eta_2 \\
\leq \int_a^b \delta\left(\tilde{g}_1(t, s, x^*(s)), \tilde{g}_1(t, s, y^*(s))\right)ds + \int_a^b \delta\left(\tilde{g}_2(t, s, x^*(s)), \tilde{g}_2(t, s, y^*(s))\right)ds + \eta_2 \\
\leq \int_a^b K \delta\left(x^*(s), y^*(s)\right)ds + \int_a^b \eta_1 ds + \eta_2.
\]

By taking the maximum for \( t \in [a, b] \), we have:
\[
\rho(x^*, y^*) \leq \max_{t \in [a, b]} \left( K \int_a^b \delta(x^*(t), y^*(t))dt + \eta_1(b-a) + \eta_2 \right) \\
= \max_{t \in [a, b]} K \delta(x^*(t), y^*(t)) \quad + \eta_1(b-a) + \eta_2 \\
= \max_{t \in [a, b]} \delta(x^*(t), y^*(t)) \quad + \eta_1(b-a) + \eta_2.
\]

Therefore,
\[
\rho(x^*, y^*) \leq \frac{\eta_2 + \eta_1(b-a)}{1 - K(b-a)}. \quad \square
\]

Consider now the question of data dependence for Volterra integral equations. We need the following metric on \( C([a, b], X) \):
\[
\rho_\star(x, y) := \max_{t \in [a, b]} \left[ \delta(x(t), y(t))e^{-\tau(t-a)} \right], \quad \text{with arbitrary } \tau > 0.
\]

The pair \((C([a, b], X), \rho_\star)\) forms a complete metric space.

It is easily seen that metrics \(\rho\) and \(\rho_\star\) satisfy the following inequalities
\[
e^{-\tau(b-a)} \rho(x, y) \leq \rho_\star(x, y) \leq \rho(x, y).
\]

We now prove the following theorem
Theorem 4.2. Let \( Y = [a, b] \times [a, b] \times X \) and let \( g_1, g_2 : Y \to X \) be weakly continuous. Consider the following equations:

\[
\begin{align*}
\quad x(t) &= f_1(t) + \int_a^t g_1(t, s, x(s))ds, \\
y(t) &= f_2(t) + \int_a^t g_2(t, s, y(s))ds.
\end{align*}
\]

Suppose:

1. For any \((t, s) \in [a, b] \times [a, b]\) function \( \tilde{g}(t, s, x) \) satisfies Lipschitz condition (3.2) on variable \( x \). (Denote by \( x^*(t) \) the unique solution of the equation (4.4)).

2. There exist \( \eta_1, \eta_2 > 0 \) such that \( \delta(\tilde{g}_1(t, s, x), \tilde{g}_2(t, s, x)) \leq \eta_1 \) for all \((t, s, x) \in [a, b] \times [a, b] \times X \) and \( \rho(f_1(t), f_2(t)) \leq \eta_2 \).

3. There exists \( y^*(t) \) a solution of the equation (4.5).

Then

\[
\rho(x^*, y^*) \leq \eta_2 + \eta_1(b - a) \left\{ \frac{1}{1 - \frac{\tau}{T}} \right\} e^{-\tau(b - a)} \quad \text{(where } \tau > K)\)
\]

and moreover

\[
\rho(x^*, y^*) \leq \eta_2 + \eta_1(b - a).
\]

Proof. We estimate

\[
\begin{align*}
\delta(x^*(t), y^*(t)) &= \delta(f_1(t) + \int_a^t \tilde{g}_1(t, s, x^*(s))ds, f_2(t) + \int_a^t \tilde{g}_2(t, s, y^*(s))ds) \\
&\leq \delta(f_1(t), f_2(t)) + \int_a^t \delta(\tilde{g}_1(t, s, x^*(s)), \tilde{g}_2(t, s, y^*(s)))ds \\
&\leq \int_a^t \delta(\tilde{g}_1(t, s, x^*(s)), \tilde{g}_1(t, s, y^*(s)))ds + \int_a^t \delta(\tilde{g}_2(t, s, y^*(s)), \tilde{g}_2(t, s, y^*(s)))ds + \eta_2 \\
&\leq \int_a^t \delta(\tilde{g}_1(t, s, x^*(s)), \tilde{g}_1(t, s, y^*(s)))ds + \int_a^t \delta(\tilde{g}(t, s, y^*(s)), \tilde{g}_2(t, s, y^*(s)))ds + \eta_2 \\
&\leq \int_a^t K \delta(x^*(s), y^*(s))e^{-\tau(s - a)}e^{\tau(s - a)}ds + \int_a^t \eta_1 ds + \eta_2.
\end{align*}
\]
By taking the maximum for \( t \in [a, b] \), we have:

\[
\max_{t \in [a, b]} (\delta(x^*(t), y^*(t))e^{-\tau(s-a)}e^{\tau(s-a)})
\]

\[
\leq \max_{t \in [a, b]} \left( K \int_a^t \delta(x^*(t), y^*(t))e^{-\tau(s-a)}e^{\tau(s-a)} dt + \int_a^t \eta_1 ds + \eta_2 \right)
\]

and therefore

\[
\rho_*(x^*, y^*)e^{\tau(b-a)} = K \rho_*(x^*, y^*) \int_a^t e^{\tau(s-a)} ds + \eta_1 (b-a) + \eta_2
\]

\[
\leq K \frac{\rho_*(x^*, y^*) e^{\tau(b-a)}}{\tau} + \eta_1 (b-a) + \eta_2.
\]

From the derived inequality, for \( \tau > K \) we obtain

\[
\rho_*(x^*, y^*) \leq \frac{\eta_2 + \eta_1 (b-a)}{1 - \frac{K}{\tau}} e^{-\tau(b-a)}.
\]

The inequality (4.6) is proved. Using the last inequality and (4.3) we have

\[
\rho(x^*, y^*) \leq e^{\tau(b-a)} \rho_*(x^*, y^*) \leq \frac{\eta_2 + \eta_1 (b-a)}{1 - \frac{K}{\tau}}.
\]

Since it is true for any \( \tau > K \) we obtain

\[
\rho(x^*, y^*) \leq \eta_2 + \eta_1 (b-a). \quad \square
\]

5. Discussion. Along with equations (3.1) and (3.5), equations of the following form are interesting:

\[
(5.1) \quad x(t) + \lambda \int_a^b g(t, s, x(s)) ds = f(t), \quad t \in [a, b],
\]

\[
(5.2) \quad x(t) + \int_a^t g(t, s, x(s)) ds = f(t), \quad t \in [a, b].
\]

Equations (5.1) and (5.2) are equivalent to equations (3.1) and (3.5) respectively in the case of real-valued functions, but not for functions with values in \( L \)-spaces.

We say that an element \( z \in X \) is the Hukuhara type difference of element \( x, y \in X \), if

\[
x = y + z.
\]
We denote this difference by
\[ z = x^h - y. \]

Difference of such type for elements of the space \( K^c(\mathbb{R}^n) \) is defined in [11] (see also [16]).

Equations (5.1) and (5.2) are equivalent to equations
\[ x(t) = f(t) - \lambda \int_a^b g(t, s, x(s))ds, \quad t \in [a, b], \]
\[ x(t) = f(t) - \int_a^t g(t, s, x(s))ds, \quad t \in [a, b]. \]

That fact that the Hukuhara type difference is not defined for all \( x \) and \( y \) brings significant difficulties into the investigation of existence and uniqueness of the solutions of these equations. We know only two references [15] and [16] in which theorems of existence and uniqueness are proved for equations of the form (5.2) in the space \( K^c(\mathbb{R}^n) \) for the special case the \( f(t) \equiv a \), where \( a \) is fixed element of \( K^c(\mathbb{R}^n) \). We will study the existence and uniqueness of solutions of (5.1) and (5.2) in future work.

REFERENCES