M1220-2 **Quiz 6** **Spring 2005**

Quiz Scores (out of 10): n = 129; mean = 6.03
25th percentile = 4.5; median (50th percentile) = 6.5; 75th percentile = 8.0

1. (5 pts) Evaluate each of the following limits:

i)

$$\lim_{x\to 0^+} \left(2x + e^{-3x}\right)^{1/x}$$

This limit is indeterminate of the form (1^{∞}).
Recall that the function $f(x) = \ln x$ is continuous so,

$$\ln\left[\lim_{x\to 0^+} \left(2x + e^{-3x}\right)^{1/x}\right] = \lim_{x\to 0^+} \left[\ln\left(2x + e^{-3x}\right)^{1/x}\right]$$

. Considering this last limit, using the properties of logs, and applying L'Hopital's Rule where appropriate we have,

$\lim_{x\to 0^+} \left[\ln\left(2x + e^{-3x}\right)^{1/x}\right] = \lim_{x\to 0^+} \left[\frac{\ln(2x+e^{-3x})}{x}\right]$ (form 0/0)

$= \lim_{x\to 0^+} \left[\frac{\frac{2-3e^{-3x}}{2x+e^{-3x}}}{1}\right] = \lim_{x\to 0^+} \left[\frac{2-3e^{-3x}}{2x+e^{-3x}}\right] = -1/1 = -1.$

Since

$$\ln\left[\lim_{x\to 0^+} \left(2x + e^{-3x}\right)^{1/x}\right] = -1$$

we have

$$\lim_{x\to 0^+} \left(2x + e^{-3x}\right)^{1/x} = e^{\ln\left[\lim_{x\to 0^+} \left(2x+e^{-3x}\right)^{1/x}\right]} = e^{-1}$$

.

ii)

$$\lim_{x\to +\infty} \left[x \sin\left(\frac{1}{x}\right)\right]$$

.

This limit is of the indeterminate form $0(\infty)$ so re-writing we have,

$$\lim_{x\to+\infty}\left[x\sin\left(\frac{1}{x}\right)\right]=\lim_{x\to+\infty}\left[\frac{\sin(\frac{1}{x})}{\frac{1}{x}}\right]\quad \text{(form } 0/0)$$

.

Applying L'Hopital's Rule we have,

$$\lim_{x\to+\infty}\left[\frac{\sin(\frac{1}{x})}{\frac{1}{x}}\right]=\lim_{x\to+\infty}\left[\frac{(\frac{-1}{x^2})\cos(\frac{1}{x})}{\frac{-1}{x^2}}\right]=\lim_{x\to+\infty}\cos(\frac{1}{x})=1$$

Comments: If you decided to write $x\sin\left(\frac{1}{x}\right)$ as $\frac{x}{\csc x}$, then in applying L'Hopital's Rule, you would get $\frac{1}{(\frac{1}{x^2})\csc x\cot x}$. Considering the limit in the denominator, you would run into another indeterminate form $(0(+\infty))$. This would need to be considered before continuing.

2. (5 pts) Evaluate the following improper integrals:

i)

$$\int_1^{+\infty}\frac{1}{\sqrt{2x+1}}dx$$

Letting $u=2x+1$ we can see this is an improper integral of the form $\int_1^{\infty}\frac{1}{x^p}dx$ where $p<1$. So this integral diverges.
If we want to examine the integral further we can do the following:

$\int\frac{1}{\sqrt{2x+1}}dx=(1/2)\int u^{-1/2}du=u^{1/2}=\sqrt{2x+1}.$

So, $\lim_{b\to+\infty}\int_1^b\frac{1}{\sqrt{2x+1}}dx=\lim_{b\to+\infty}\left[\sqrt{2b+1}-\sqrt{3}\right]=+\infty.$

ii)

$$\int_2^{+\infty}xe^{-x^2}dx$$

Letting $u = x^2, du = 2xdx$ and noting that when $x = 2, u = 4$ and when $x \to +\infty, u \to +\infty$, we can rewrite the integral as $(1/2)\int_4^{+\infty} e^{-u}du$. Then, $(1/2)\int_4^{+\infty} e^{-u}du = (1/2)\lim_{b\to+\infty}\int_4^b e^{-u}du$
$= (1/2)\lim_{b\to+\infty}\left[-e^{-u}|_4^b\right] = (1/2)\lim_{b\to+\infty}\left[-e^{-b} + e^{-4}\right] = (1/2)e^{-4}$.

Comments: Note that if you let $u = -x^2, du = -2xdx$ then in converting to the integral in terms of u, the integration limits change so that when $x = 2, u = -4$, and when $x \to +\infty, u \to -\infty$.